1)

Notes for 554 Paper, Part 5

The Coriolis and Centripetal Accelerations

Classical Background

(J. B. Marion and S. T. Thornton "Classical Dynamics (HBC, NY, 1988, chapter 9)

Consider two sets of coordinate axes. One is fixed or inertial and the other is in arbitrary motion with respect to the first. These are designated "fixed" and "rotating"

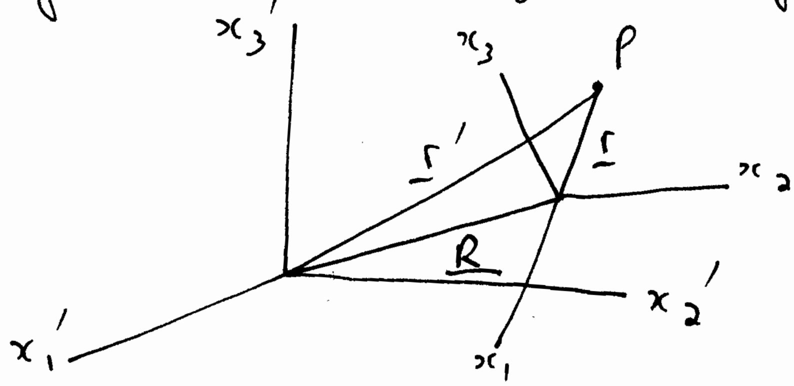

Fig (1).

For any point P:

$$\underset{\sim}{r}' = \underset{\sim}{R} + \underset{\sim}{r} \quad -(1)$$

as in Fig (1). It is always possible to represent an arbitrary infinitesimal displacement by a pure rotation about an axis called the instantaneous axis of rotation. For example, if a disk ~~rolls~~ rolls down an inclined plane the motion is a rotation about the point of contact of the disk with the plane. Therefore if the x_i system undergoes an infinitesimal rotation $d\theta$, corresponding to an arbitrary infinitesimal displacement,

$$\left(d\underline{r}\right)_{\text{fixed frame}} = d\underline{\theta} \times \underline{r} \quad -(2)$$

2) Eq. (2) is the result of geometry. The quantity $\underline{dr}$ is measured in the x_i', or fixed, coordinate system. The point P is considered to be at rest with respect to the x_i system, but P is moving with respect to the x_i' system.

Now divide eq. (2) by the time interval during which the infinitesimal rotation takes place:

$$\left(\frac{d\underline{r}}{dt}\right)_{\text{fixed frame}} = \frac{d\underline{\theta}}{dt} \times \underline{r}. \quad -(3)$$

Now identify the angular velocity as:

$$\underline{\omega} = \frac{d\underline{\theta}}{dt}. \quad -(4)$$

So:

$$\left(\frac{d\underline{r}}{dt}\right)_{\text{fixed frame}} = \underline{\omega} \times \underline{r}. \quad -(5)$$

Now consider $\underline{P}$ to have a velocity with respect to the x_i system:

$$\left(\underline{v} = \frac{d\underline{r}}{dt}\right)_{\text{rotating frame}} \quad -(6)$$

The total velocity is:

$$\boxed{\left(\frac{d\underline{r}}{dt}\right)_{\text{fixed frame}} = \left(\frac{d\underline{r}}{dt}\right)_{\text{rotating frame}} + \underline{\omega} \times \underline{r}}$$

(7)

3) Eq. (7) is valid for an arbitrary vector $\underline{Q}$:

$$\left(\frac{d\underline{Q}}{dt}\right)_{\text{fixed frame}} = \left(\frac{d\underline{Q}}{dt}\right)_{\text{rotating frame}} + \underline{\omega} \times \underline{Q} \qquad -(8)$$

If $\underline{Q}$ is linear velocity $\underline{v}$ ~~vec~~ :

$$\left(\frac{d\underline{v}}{dt}\right)_{\text{fixed frame}} = \left(\frac{d\underline{v}}{dt}\right)_{\text{rotating frame}} + \underline{\omega} \times \underline{v} \qquad -(9)$$

where from eq. (7) :

$$\underline{v}_{\text{fixed frame}} = \underline{v}_{\text{rotating frame}} + \underline{\omega} \times \underline{r} \qquad -(10)$$

Differentiate eq. (10) :

$$\left(\frac{d\underline{v}}{dt}\right)_{\text{fixed frame}} = \left(\frac{d\underline{v}}{dt}\right)_{\text{rotating frame}} + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \frac{d\underline{r}}{dt} \qquad -(11)$$

from eq. (7) :

$$\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{\text{fixed frame}} = \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{\text{rotating frame}} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \qquad -(12)$$

4) The quantity $\underline{\omega} \times \underline{v}$ is part of the Coriolis acceleration and $\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is the negative of the centripetal acceleration.

Standard Definitions

The standard definitions are found by writing eq. (8) as:

$$\left(\frac{d\underline{Q}}{dt}\right)_{\text{rotating frame}} = \left(\frac{d\underline{Q}}{dt}\right)_{\text{fixed frame}} - \underline{\omega} \times \underline{Q} \quad -(13)$$

The Coriolis acceleration was inferred in 1835 by G. G. Coriolis, who considered an observer in the rotating coordinate system. The standard definition of the Coriolis acceleration is obtained as follows (Marion and Thornton, pp. 359 ff). First differentiate eq. (1):

$$\left(\frac{d\underline{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\underline{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\underline{r}}{dt}\right)_{\text{fixed}} \quad -(14)$$

$$= \left(\frac{d\underline{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\underline{r}}{dt}\right)_{\text{rotating}} + \underline{\omega} \times \underline{r} \quad -(15)$$

Now define:

5)

$$\underline{v}_f = \dot{\underline{r}}_f = \left(\frac{d\underline{r}'}{dt}\right)_{fixed} \quad -(16)$$

$$\underline{V} = \dot{\underline{R}}_f = \left(\frac{d\underline{R}}{dt}\right)_{fixed} \quad -(17)$$

$$\underline{v}_r = \dot{\underline{r}}_f = \left(\frac{d\underline{r}}{dt}\right)_{rotating} \quad -(18)$$

so:

$$\underline{v}_f = \underline{V} + \underline{v}_r + \underline{\omega} \times \underline{r} \quad -(19)$$

The Newtonian acceleration is ~~the~~ term:

$$\underline{F} = m\left(\frac{d\underline{v}_f}{dt}\right)_{fixed} . \quad -(20)$$

However, differentiation of eq. (19) gives:

$$\left(\frac{d\underline{v}_f}{dt}\right)_{fixed} = \left(\frac{d\underline{V}}{dt}\right)_{fixed} + \left(\frac{d\underline{v}_r}{dt}\right)_{fixed} \quad -(21)$$
$$+ \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{fixed}$$

Denote:

$$\ddot{\underline{R}}_f = \left(\frac{d\underline{V}}{dt}\right)_{fixed} \quad -(22)$$

The second term in eq. (21) is evaluated by substituting $\underline{v}_r$ for $\underline{Q}$ in eq. (13):

$$\left(\frac{d\underline{v}_r}{dt}\right)_{fixed} = \left(\frac{d\underline{v}_r}{dt}\right)_{rotating} + \underline{\omega} \times \underline{v}_r$$

$$= \underline{a}_r + \underline{\omega} \times \underline{v}_r \quad -(23)$$

The last term in eq. (21) is obtained from:

$$\underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{fixed} = \underline{\omega} \times \left(\frac{d\underline{r}}{dt}\right)_{rotating} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(24)$$

$$= \underline{\omega} \times \underline{v}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

So:

$$\underline{F} = m\underline{a}_f = m\underline{\ddot{R}}_f + m\underline{a}_r + m\underline{\dot{\omega}} \times \underline{r} + m\underline{\omega} \times (\underline{\omega} \times \underline{r}) + 2m\underline{\omega} \times \underline{v}_r. \quad -(25)$$

To an observer in the rotating coordinate system:

$$\boxed{\underline{F}_r = m\underline{a}_r = \underline{F} - m\underline{\ddot{R}}_f - m\underline{\dot{\omega}} \times \underline{r} - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) - 2m\underline{\omega} \times \underline{v}_r} \quad -(26)$$

The centripetal force is $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$, and is directed outward from the center of rotation.

The Coriolis force is $-2m\underline{\omega} \times \underline{v}_r$.

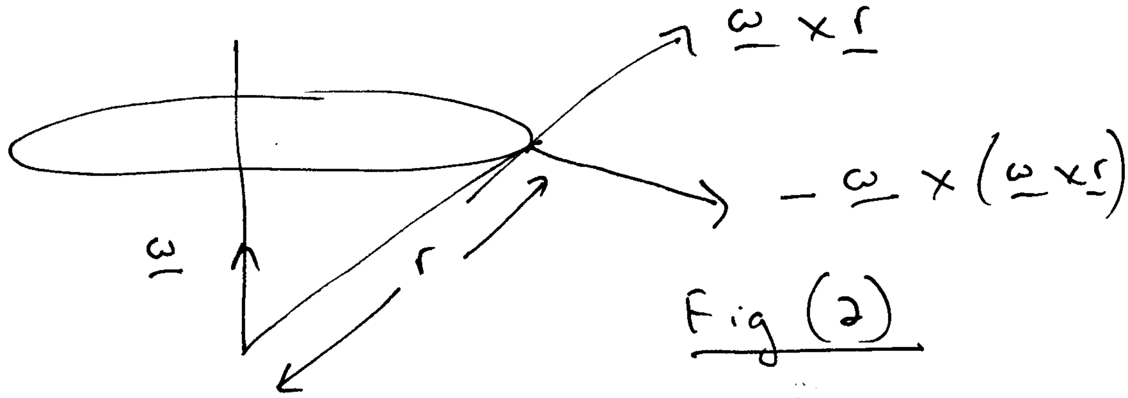

Fig (2)

Thus:

$$\underline{F}_r = m\underline{a}_f + \text{non-inertial terms} \quad -(27)$$

Generally Covariant Treatment

The classical theory is not generally covariant because it uses Euclidean space-time. The generally covariant treatment needs ECE space-time with torsion and curvature. The Cartan torsion is:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(28)$$

where q^a is the tetrad and $\omega^a{}_b$ is the spin connection. Eq. (28) is the Cartan structure equation. The first Bianchi identity is:

$$d \wedge T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge q^b \quad -(29)$$

where $R^a{}_b$ is the Riemann form. In an inertial or central theory such as that of Einstein and Hilbert:

$$R^a{}_b \wedge q^b = 0 \quad -(30)$$

$$T^a = 0 \quad -(31)$$

8) However in a non-vertical theory:

$$\boxed{\begin{aligned} R \wedge q &= d \wedge (d \wedge q + \omega \wedge q) \\ &+ \omega \wedge (d \wedge q + \omega \wedge q) \end{aligned}} \quad -(32)$$

whereas in an vertical theory:

$$R \wedge q = 0 \quad -(33)$$

The terms in eq. (32) come from the rotation of space-time itself.

We may always define:

$$Q = Q^{(0)} q \quad -(34)$$

where Q is the covariant generalization of the vector $\underline{Q}$ in eq. (8). For example there is a position tetrad and a velocity tetrad:

$$r = r^{(0)} q \quad -(35)$$

$$v = v^{(0)} q \quad -(36)$$

So:

$$\begin{aligned} R \wedge r &= d \wedge (d \wedge r + \omega \wedge r) \\ &+ \omega \wedge (d \wedge r + \omega \wedge r) \end{aligned} \quad -(37)$$

$$\begin{aligned} R \wedge v &= d \wedge (d \wedge v + \omega \wedge v) \\ &+ \omega \wedge (d \wedge v + \omega \wedge v) \end{aligned} \quad -(38)$$

From eq. (37) we find the term:

$$\boxed{(R \wedge r)_{centripetal} = \omega \wedge (\omega \wedge r)} \quad -(39)$$

9) and from eq. (38) the term:

$$\boxed{(R \wedge v)_{Coriolis} = d \wedge (\omega \wedge v) + \omega \wedge (\omega \wedge v}$$

— (40) —

Conclusion

1) The centripetal acceleration originates in $\omega \wedge (\omega \wedge r)$ and the Coriolis acceleration $\omega \wedge v$.

2) The Euler equation originates in the Cartan torsion itself:

$$\boxed{T = d \wedge q + \omega \wedge q} . \quad — (41)$$

When there is no space-time spin:

$$\omega = 0, \quad — (42)$$

and

$$\omega \wedge (\omega \wedge r) = 0, \quad — (43)$$

$$\omega \wedge v = 0, \quad — (44)$$

$$d \wedge q = 0, \quad — (45)$$

and so:

$$d \wedge v = d \wedge r = 0, \quad — (46)$$

$$T = 0 . \quad — (47)$$

In this limit the Newtonian force is recovered from a limiting procedure based on the second Bianchi identity:

$$D \wedge \omega = 0, \quad — (48)$$

which gives the Einstein Hilbert field equation