Notes for Paper 55 Part 8

ECE Field Equations for Gravitation (Translational Motion)

In analogy with papers 52 and 53, and notes 55(2), equations of motion may be constructed for gravitation and translational dynamics. In paper 52 the electromagnetic field tensor was defined as:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{bmatrix}, \quad -(1)$$

and the electric and magnetic fields as:

$$\underline{E}^a = -\frac{\partial \underline{A}^a}{\partial t} - c\underline{\nabla} A^{0a} - c\,\omega^{0a}{}_b \underline{A}^b + c\,\underline{\omega}^a{}_b A^{0b}, \quad -(2)$$

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b. \quad -(3)$$

In notes 55(2) the Cartan torsion tensor was defined in the same way as eq. (1) and the orbital and spin torsion vectors defined as follows via these tensor components:

2)

$$T^{\mu\nu} = \begin{bmatrix} 0 & -T_L^1/c & -T_L^2/c & -T_L^3/c \\ T_L^1/c & 0 & -T_S^3 & T_S^2 \\ T_L^2/c & T_S^3 & 0 & -T_S^1 \\ T_L^3/c & -T_S^2 & T_S^1 & 0 \end{bmatrix} \quad -(4)$$

where: T_L = orbital torsion,
T_S = spin torsion.

Therefore:

$$\underline{T}_L^{\,a} = -\frac{\partial \underline{q}^a}{\partial t} - c\underline{\nabla} q^{0a} - c\omega^{0a}{}_b\,\underline{q}^b + c\underline{\omega}^a{}_b\, q^{0b}, \quad -(5)$$

$$\underline{T}_S^{\,a} = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b. \quad -(6)$$

This procedure defined the orbital torque:

$$\underline{N}_L^{\,a} = -\frac{\partial \underline{J}^a}{\partial t} - c\underline{\nabla} J^{0a} - c\omega^{0a}{}_b\,\underline{J}^b + c\underline{\omega}^a{}_b\, J^{0b}, \quad -(7)$$

and the spin torque:

$$\underline{N}_S^{\,a} = c\left(\underline{\nabla} \times \underline{J}^a - \underline{\omega}^a{}_b \times \underline{J}^b\right). \quad -(8)$$

3) The orbital torque is analogous to the electric field strength (cf. eqns. (7) and (2)). The spin torque is analogous to the magnetic flux density (cf. eqns (8) and (3)).

This analogy can now be taken one stage further using notes 55(6) and 55(7), where the orbital Riemann vector was defined as:

$$\underline{R}^a{}_b(\text{orbital}) = R^0{}_{101}\,\underline{i} + R^0{}_{202}\,\underline{j} + R^0{}_{303}\,\underline{k}, \quad -(9)$$

and the spin Riemann vector as:

$$\underline{R}^a{}_b(\text{spin}) = R^2{}_{323}\,\underline{i} + R^1{}_{331}\,\underline{j} + R^1{}_{212}\,\underline{k}. \quad -(10)$$

Therefore the analogue of eqns (1) and (4) is set up as follows using the same units (orbital components divided by a factor of c):

$$R^a{}_{b\mu\nu} = \begin{bmatrix} 0 & -R^0{}_{101}/c & -R^0{}_{202}/c & -R^0{}_{303}/c \\ R^0{}_{101}/c & 0 & -R^1{}_{212} & R^1{}_{313} \\ R^0{}_{202}/c & R^1{}_{221} & 0 & -R^2{}_{323} \\ R^0{}_{303}/c & -R^1{}_{331} & R^2{}_{332} & 0 \end{bmatrix} \quad -(11)$$

4) In analogy with eqns (2), (3), (5) and (6), the orbital and spin Riemann components are defined by the second Cartan structure equation:

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \quad -(9)$$

In vector notation, eqn. (9) is:

$$\underline{R}^a{}_b(\text{orbital}) = -\frac{\partial \underline{\omega}^a{}_b}{\partial t} - c\underline{\nabla}\,\omega^{0a}{}_b - c\,\omega^{0a}{}_c\,\underline{\omega}^c{}_b + c\,\underline{\omega}^a{}_c\,\omega^{0c}{}_a \quad -(10)$$

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_b \times \underline{\omega}^b{}_c. \quad -(11)$$

The conventional definition of the Riemann tensor in Riemann geometry is:

$$R^\rho{}_{\sigma\mu\nu} = R^a{}_{b\mu\nu}\, q^b{}_\sigma\, q^\rho{}_a \quad -(12)$$

$$= \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu R^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \quad -(13)$$

and this definition includes equations (10) and (11). However it is not clear from eqns (12) and (13) that these exist orbital and spin components.

5) ## The Field Equations

The field equations of electrodynamics, and of rotational and translational motion, are all given by the first and second Bianchi identities:

$$D \wedge T^a := d \wedge T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge q^b, \quad -(14)$$

$$D \wedge R^a{}_b = 0. \quad -(15)$$

Field Equations of Electrodynamics

Form Notation

$$d \wedge F^a = \mu_0 j^a \quad -(16)$$

$$d \wedge \tilde{F}^a = \mu_0 J^a \quad -(17)$$

where:

$$j^a = \frac{A^{(0)}}{\mu_0} \left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right), \quad -(18)$$

$$J^a = \frac{A^{(0)}}{\mu_0} \left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b \right). \quad -(19)$$

Tensor Notation

$$\partial_\mu \tilde{F}^{a\mu\nu} = \mu_0 \tilde{j}^{a\nu} \quad -(20)$$

$$\partial_\mu F^{a\mu\nu} = \mu_0 \tilde{J}^{a\nu} \quad -(21)$$

where:

$$\tilde{j}^{a\nu} = \frac{A^{(0)}}{\mu_0}\left(\tilde{R}^{a}{}_{\mu}{}^{\mu\nu} - \omega^{a}{}_{\mu b}\tilde{T}^{b\mu\nu}\right) \quad -(22)$$

$$\tilde{J}^{a\nu} = \frac{A^{(0)}}{\mu_0}\left(R^{a}{}_{\mu}{}^{\mu\nu} - \omega^{a}{}_{\mu b}T^{b\mu\nu}\right). \quad -(23)$$

<u>Vector Notation</u>

Define:

$$\tilde{j}^{a\nu} = \left(\frac{1}{c}\tilde{j}^{a0}, \underline{\tilde{j}}^{a}\right) \quad -(24)$$

$$\tilde{J}^{a\nu} = \left(\frac{1}{c}\tilde{J}^{a0}, \underline{\tilde{J}}^{a}\right). \quad -(25)$$

Then:

$$\underline{\nabla}\cdot\underline{B}^{a} = \mu_0\,\tilde{j}^{a0} \quad -(26)$$

$$\underline{\nabla}\times\underline{E}^{a} + \frac{\partial\underline{B}^{a}}{\partial t} = \mu_0\,\underline{\tilde{j}}^{a} \quad -(27)$$

$$\underline{\nabla}\cdot\underline{E}^{a} = \mu_0 c\,\tilde{J}^{a0} \quad -(28)$$

$$\underline{\nabla}\times\underline{B}^{a} - \frac{1}{c^2}\frac{\partial\underline{E}^{a}}{\partial t} = \frac{\mu_0}{c}\,\underline{\tilde{J}}^{a}. \quad -(29)$$

In direct analogy the equations for rotational and translational dynamics are as follows.

7) Field Equations of Rotational Dynamics

Form Notation

$$\boxed{\begin{aligned} d \wedge T^a &= j^a \\ d \wedge \tilde{T}^a &= \tilde{J}^a \end{aligned}}$$

— (30)
— (31)

where

$$j^a = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \quad — (32)$$

$$J^a = \tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b . \quad — (33)$$

Tensor Notation

$$\boxed{\begin{aligned} \partial_\mu \tilde{T}^{a\mu\nu} &= \tilde{j}^{a\nu} \\ \partial_\mu T^{a\mu\nu} &= \tilde{J}^{a\nu} \end{aligned}}$$

— (34)
— (35)

where:

$$\tilde{j}^{a\nu} = \tilde{R}^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} \tilde{T}^{b\mu\nu} \quad — (36)$$

$$\tilde{J}^{a\nu} = R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} T^{b\mu\nu} \quad — (37)$$

Vector Notation

Eqs (24) and (25) remain the same, so the four equations of rotational dynamics are the field equations:

8)

$$\boxed{\begin{aligned}
\underline{\nabla} \cdot \underline{T}_s^{\;a} &= \tilde{j}^{\,a0} / c \qquad -(38)\\
\underline{\nabla} \times \underset{\sim}{T}_L^{\;a} + \frac{1}{c}\frac{\partial \underset{\sim}{T}_s^{\;a}}{\partial t} &= \underline{\tilde{j}}^{\,a} \qquad -(39)\\
\underline{\nabla} \cdot \underset{\sim}{T}_L^{\;a} &= \tilde{J}^{\,a0} \qquad -(40)\\
\underline{\nabla} \times \underset{\sim}{T}_s^{\;a} - \frac{1}{c^2}\frac{\partial \underset{\sim}{T}_L^{\;a}}{\partial t} &= \underline{\tilde{J}}^{\,a} \qquad -(41)
\end{aligned}}$$

Field Equations of Translational Dynamics

Form Notation

$$D \wedge R^a{}_b = d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \qquad -(42)$$

This is the second Bianchi identity. So eq. (42) is:

$$\boxed{\begin{aligned}
d \wedge R^a{}_b &= j^a{}_b \qquad -(43)\\
d \wedge \tilde{R}^a{}_b &= J^a{}_b \qquad -(44)
\end{aligned}}$$

where:

$$j^a{}_b = R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b$$

$$J^a{}_b = \tilde{R}^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \tilde{R}^c{}_b$$

9)

Tensor Notation

$$\boxed{\begin{aligned} \partial_\mu \tilde{R}^{a}{}_{b}{}^{\mu\nu} &= \tilde{j}^{a}{}_{b}{}^{\nu} \quad &-(45) \\ \partial_\mu R^{a}{}_{b}{}^{\mu\nu} &= \tilde{J}^{a}{}_{b}{}^{\nu} \quad &-(46) \end{aligned}}$$

where:

$$\tilde{j}^{a}{}_{b}{}^{\nu} = \tilde{R}^{a}{}_{c}{}^{\mu\nu}\omega^{c}{}_{\mu b} - \omega^{a}{}_{\mu c}\tilde{R}^{c}{}_{b}{}^{\mu\nu} \quad -(47)$$

$$\tilde{J}^{a}{}_{b}{}^{\nu} = R^{a}{}_{c}{}^{\mu\nu}\omega^{c}{}_{\mu b} - \omega^{a}{}_{\mu c}R^{c}{}_{b}{}^{\mu\nu} \quad -(48)$$

Vector Notation

Eqs (24) and (25) remain the same and:

$$\boxed{\begin{aligned} \underline{\nabla}\cdot\underline{R}_S^{a} &= \tilde{j}^{a0}/c \quad &-(49) \\ \underline{\nabla}\times\underline{R}_L^{a} + \frac{1}{c}\frac{\partial \underline{R}_S^{a}}{\partial t} &= \underline{\tilde{j}}^{a} \quad &-(50) \\ \underline{\nabla}\cdot\underline{R}_L^{a} &= \tilde{J}^{a0} \quad &-(51) \\ \underline{\nabla}\times\underline{R}_S^{a} - \frac{1}{c^2}\frac{\partial \underline{R}_L^{a}}{\partial t} &= \underline{\tilde{J}}^{a} \quad &-(52) \end{aligned}}$$

10) In general these equations are for mixed rotation and translation. Newtonian dynamics emerge from eq. (51) for pure translation, which is defined by

$$\boxed{\begin{aligned} R^a{}_b &= D \wedge \omega^a{}_b && (53)\\ D \wedge R^a{}_b &= 0 && (54)\\ T^a &= 0 && (55)\\ R^a{}_b \wedge q^b &= 0 && (56) \end{aligned}}$$

Pure Rotation is defined by:

$$\boxed{\begin{aligned} T^a &= D \wedge q^a && (57)\\ d \wedge T^a &= 0 && (58)\\ R^a{}_b \wedge q^b &= \omega^a{}_b \wedge T^b && (59) \end{aligned}}$$

Mixed rotation / translation is defined by:

$$\boxed{\begin{aligned} T^a &= D \wedge q^a && (60)\\ R^a{}_b &= D \wedge \omega^a{}_b && (61)\\ D \wedge T^a &= R^a{}_b \wedge q^b && (62)\\ D \wedge R^a{}_b &= 0 && (63) \end{aligned}}$$

These equations also define the influence of gravitation on electromagnetism