APPENDIX : ANALYTICAL SOLUTIONS OF THE UNDAMPED OSCILLATOR.

Consider the basic undamped oscillator equation (see text):

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0} f(\kappa R) \quad \text{—(A1)}$$

where:

$$f(\kappa R) = e^{2i\kappa R} \cos\left(e^{i\kappa R}\right) \quad \text{—(A2)}$$

If $f(\kappa R)$ Satisfies the Dirichlet condition, i.e. is single valued and continuous in an interval such as $\pi < f(\kappa R) \leq \pi$ it can be expanded in a Fourier series:

$$f(\kappa R) = \frac{a_0}{2} + \sum_{d=1}^{\infty} \left(a_d \cos(d\kappa R) + b_d \sin(d\kappa R) \right) \quad \text{—(A3)}$$

where:

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\kappa R)\, d(\kappa R) \\ a_d &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\kappa R)\cos(d\kappa R)\, d(\kappa R) \\ b_d &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\kappa R)\sin(d\kappa R)\, d(\kappa R). \end{aligned}\right\} \quad \text{—(A4)}$$

These integrals can be computed straightforwardly to any required precision in any interval, the latter is not necessarily constrained to $\pi < f(\kappa R) \leq \pi$, the latter is used for illustration. Therefore eq. (A1) becomes:

$$\frac{d^2\phi}{dR^2} + \kappa^2 R = \frac{\rho(0)}{\epsilon_0}\left(\frac{a_0}{2} + a_1\cos(\kappa R) + a_2\cos(2\kappa R) + \ldots + b_1 \sin(\kappa R) + b_2\sin(2\kappa R) + \ldots\ldots \right) \quad \text{—(A5)}$$

Assume a solution of the type:

$$\phi = \frac{\rho(0)}{\epsilon_0}\left(A_0 \frac{a_0}{2} + A_1 a_1 \cos(\kappa R) + A_2 a_2 \cos(2\kappa R) + \ldots + B_1 b_1 \sin(\kappa R) + B_2 b_2 \sin(2\kappa R) + \ldots\right). \quad -(A6)$$

Substituting Eq. (A6) in Eq. (A5) and comparing terms by term:

$$\left.\begin{aligned} \kappa^2 A_0 \frac{a_0}{2} &= \frac{a_0}{2} \\ A_1 \kappa^2 (1 - a_1) \cos(\kappa R) &= \cos(\kappa R) \\ A_2 \kappa^2 (1 - 4a_2) \cos(\kappa R) &= \cos(\kappa R) \\ &\vdots \\ B_n \kappa^2 (1 - n^2 b_n) \sin(n\kappa R) &= \sin(n\kappa R). \end{aligned}\right] \quad -(A7)$$

Thus:

$$\phi = \frac{\rho(0)}{\epsilon_0 \kappa^2}\left(\frac{a_0}{2} + \frac{\cos(\kappa R)}{(1-a_1)} + \frac{\cos(2\kappa R)}{(1-4a_2)} + \ldots + \frac{\sin(\kappa R)}{(1-b_1)} + \frac{\sin(2\kappa R)}{(1-4b_2)} + \ldots\right). \quad -(A8)$$

Infinite resonances occur at:

$$\left.\begin{aligned} a_n &= 1/n^2, \; n = 1, \ldots, m, \\ b_n &= 1/n^2, \; n = 1, \ldots, m. \end{aligned}\right\} \quad -(A9)$$

$$-(A10)$$

In general these resonances occur at:

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R} \cos\left(e^{i\kappa R}\right) \cos(n\kappa R)\, d(\kappa R)\right) = \frac{1}{n^2}$$

and:

$$\mathrm{Real}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2i\kappa R}\cos\left(e^{i\kappa R}\right)\sin(2\kappa R)\,d(\kappa R)\right) = \frac{1}{n^2}. \qquad -(A11)$$

This analysis can be repeated straightforwardly for any driving term:

$$f(\kappa R) = e^{2i\kappa R} f_1\left(e^{i\kappa R}\right). \qquad -(A12)$$

A constrained particular integral of Eq. (A1) can be obtained for any driving function $f(\kappa R)$. In this case the undamped oscillator is:

$$\frac{d^2\phi}{dR^2} + \kappa^2\phi = \frac{\rho(0)}{\epsilon_0} e^{2i\kappa R} f\left(e^{i\kappa R}\right). \qquad -(A13)$$

Assume a solution of the type:

$$\phi = \frac{A\rho(0)}{\epsilon_0} e^{2i\kappa R} f\left(e^{i\kappa R}\right) \qquad -(A14)$$

subject to the constraint:

$$\frac{dA}{dR} = 0. \qquad -(A15)$$

Then:

$$\frac{d\phi}{dR} = i\kappa \frac{A\rho(0)}{\epsilon_0}\left(2e^{2i\kappa R} f + e^{3i\kappa R} f'\right) \qquad -(A16)$$

and:

$$\frac{d^2\phi}{dR^2} = -\kappa^2 \frac{A\rho(0)}{\epsilon_0}\left(4e^{2i\kappa R} f + 5e^{3i\kappa R} f' + e^{4i\kappa R} f''\right). \qquad -(A17)$$

So the particular integral is:

$$\phi = -\frac{\rho(0)}{\epsilon_0}\left(\frac{r^2 f^2}{3f + 5\kappa r f' + \kappa^2 r^2 f''}\right) \quad \text{—(A18)}$$

subject to the constraint:

$$\frac{d\phi}{dr} = 0. \quad \text{—(A19)}$$

A solution of Eq. (A19) is the general resonance condition:

$$3f + 5\kappa r f' + \kappa^2 r^2 f'' = 0. \quad \text{—(A20)}$$

To explain the notation in Eq. (A20) consider for example a cosine driving term:

$$f = \cos x, \quad x = \kappa r. \quad \text{—(A21)}$$

Then the notation means:

$$f' = -\sin x, \quad f'' = \cos x. \quad \text{—(A22)}$$

The resonance condition (A20) then becomes:

$$\tan x = \frac{3 - x^2}{5x} \quad \text{—(A23)}$$

to which there is an infinite number of solutions. For a driving term:

$$f = e^{-x}, \quad f' = -e^{-x}, \quad f'' = e^{-x} \quad \text{—(A24)}$$

the resonance condition is:

$$x^2 - 5x + 3 = 0 \quad \text{— (A25)}$$

and there are two solutions at

$$x = 4.3028, \; 0.6972. \quad \text{— (A26)}$$

For a driving term:

$$\left.\begin{aligned} f &= e^{-x} \cos x \\ f' &= e^{-x}(\cos x - \sin x) \\ f'' &= -2e^{-x} \sin x \end{aligned}\right] \quad \text{— (A27)}$$

the resonance equation is:

$$\tan x = \frac{3 + 5x}{x(5 - 6x)} \quad \text{— (A28)}$$

and there are again an infinite number of solutions.