1/63(4): Spin Connection for Hartree Potential

In this case:

$$\underline{E} = -\underline{\nabla}\phi = \underline{\omega}\phi \quad -(1)$$

where:

$$\underline{E}(\underline{r}) = -\frac{1}{4\pi\epsilon_0}\underline{\nabla}\int\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}d^3r' \quad -(2)$$

and

$$\phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}d^3r' \quad -(3)$$

From eq. (1):

$$-\int\underline{\nabla}\phi\, d^3r' = \int\underline{\omega}\phi\, d^3r' \quad -(4)$$

where:

$$-\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) = \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} \quad -(5)$$

so:

$$\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} = \frac{\underline{\omega}}{|\underline{r}-\underline{r}'|} \quad -(6)$$

$$\boxed{\underline{\omega} = \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^2}} \quad -(7)$$

2) The electric field is defined by:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(8)$$

$$\underline{E} = -\underline{\nabla} \phi + \underline{\omega} \phi \quad -(9)$$

So the resonance equation is:

$$\underline{\nabla} \cdot \left(-\underline{\nabla} \phi + \underline{\omega} \phi \right) = \rho / \epsilon_0 \quad -(10)$$

i.e.

$$\nabla^2 \phi - \underline{\nabla} \cdot (\underline{\omega} \phi) = -\rho / \epsilon_0 \quad -(11)$$

$$\nabla^2 \phi - \underline{\omega} \cdot \underline{\nabla} \phi - (\underline{\nabla} \cdot \underline{\omega}) \phi = -\rho / \epsilon_0 \quad -(12)$$

In the Z direction:

$$\frac{\partial^2 \phi}{\partial Z^2} - \omega_Z \frac{\partial \phi}{\partial Z} - \frac{\partial \omega_Z}{\partial Z} \phi = -\rho / \epsilon_0 \quad -(13)$$

where:

$$\omega_Z = \frac{Z - Z'}{(Z - Z')^2} = \frac{1}{Z - Z'}$$

So:

$$\frac{\partial^2 \phi}{\partial Z^2} - \frac{1}{Z - Z'} \frac{\partial \phi}{\partial Z} - \frac{\partial}{\partial Z} \left(\frac{1}{Z - Z'} \right) \phi = -\rho / \epsilon_0 \quad -(14)$$

3) If there are n charges:

$$\nabla^2\phi - \underline{\omega}\cdot\underline{\nabla}\phi - (\underline{\nabla}\cdot\underline{\omega})\phi = -\frac{\rho}{\epsilon_0} \quad -(15)$$

where:

$$\underline{\omega} = \frac{\underset{\sim}{r} - \underset{\sim}{r}'}{|\underset{\sim}{r} - \underset{\sim}{r}'|^2} \quad -(16)$$

and

$$\rho(\underset{\sim}{r}) = \sum_{i=1}^{n} q_i\,\delta(\underset{\sim}{r} - \underset{\sim}{r}_i) \quad -(17)$$

where $\delta(\underset{\sim}{r} - \underset{\sim}{r}_i)$ is the Dirac delta function.

The electric field is:

$$\underline{E}(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i \frac{\underset{\sim}{r} - \underset{\sim}{r}_i}{|\underset{\sim}{r} - \underset{\sim}{r}_i|^3} \quad -(18)$$

$$= \frac{1}{4\pi\epsilon_0}\int \rho(\underset{\sim}{r}')\frac{(\underset{\sim}{r} - \underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|^3}\,d^3r'$$

If Δq is the charge in a small volume

$$d^3r = \Delta x\,\Delta y\,\Delta z \quad -(19)$$

then

$$\Delta q = \rho(\underset{\sim}{r}')\,\Delta x\,\Delta y\,\Delta z \quad -(20)$$

4)

So:

$$\boxed{\nabla^2\phi - \frac{(\underset{\sim}{r}-\underset{\sim}{r}_i)}{|\underset{\sim}{r}-\underset{\sim}{r}_i|^2}\cdot\underline{\nabla}\phi - \left(\underline{\nabla}\cdot\frac{(\underset{\sim}{r}-\underset{\sim}{r}_i)}{|\underset{\sim}{r}-\underset{\sim}{r}_i|^2}\right)\phi = -\frac{1}{\epsilon_0}\left(\sum_{i=1}^{n} q_i\,\delta(\underset{\sim}{r}-\underset{\sim}{r}_i)\right)} \quad -(21)$$

where the spin connection has been defined by:

$$\boxed{\underline{\omega}_i = \frac{\underset{\sim}{r}-\underset{\sim}{r}_i}{|\underset{\sim}{r}-\underset{\sim}{r}_i|^2}} \quad -(22)$$

This describes a system of point charges q_i located at $\underset{\sim}{r}_i$, $i = 1, \ldots, n$.