70(3): Relation Between Pauli Spinors and Tetrad, the Chiral Elements

In note 70(1) it was shown that the tetrad for the fermion field is:

$$q^a_\mu = \begin{bmatrix} q^R_1 & q^R_2 \\ q^L_1 & q^L_2 \end{bmatrix} \quad -(1)$$

and is defined by the chirality column vector $\bar{V}^a$ and the spin column vector $\bar{V}^\mu$ as follows:

$$\bar{V}^a = q^a_\mu \bar{V}^\mu. \quad -(2)$$

A simple example of $\bar{V}^\mu$ is the column vector:

$$\bar{V}^\mu = e^{-i\phi} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(3)$$

and the chirality column vector is:

$$\bar{V}^a = \frac{1}{\sqrt{2}} \begin{bmatrix} 1-i \\ 1+i \end{bmatrix} \quad -(4)$$

The right and left handed spins are the most fundamental elements of the fermion field. The well known Pauli spinors are constructed from the tetrad as follows:

$$(\phi^R)^T = [1 \quad 0]\, q^a_\mu \quad -(5)$$

$$(\phi^L)^T = [0 \quad 1]\, q^a_\mu \quad -(6)$$

where T denotes "transpose".

2) Thus:

$$(\phi^R)^T = [1 \;\; 0]\begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} = [q_1^R \;\; q_2^R]$$

and
$$\phi^R = \begin{bmatrix} q_1^R \\ q_2^R \end{bmatrix}. \quad -(7)$$

Similarly:

$$(\phi^L)^T = [0 \;\; 1]\begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} = [q_1^L \;\; q_2^L]$$

and
$$\phi^L = \begin{bmatrix} q_1^L \\ q_2^L \end{bmatrix}. \quad -(8)$$

Thus:
$$\phi^R = \left([1, 0]\, q_\mu^a\right)^T \quad -(9)$$
$$\phi^L = \left([0, 1]\, q_\mu^a\right)^T \quad -(10)$$

and the Dirac spinor is:
$$\psi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix}. \quad -(11)$$

It follows from the ECE wave equation:
$$(\Box + kT)\, q_\mu^a = 0 \quad -(12)$$

that:
$$(\Box + kT)\, \psi = 0 \quad -(13)$$

3)

In the absence of fields other than the fermion field, i.e. for a free fermion:

$$kT \rightarrow \left(\frac{mc}{\hbar}\right)^2 \quad -(14)$$

and we recover the Dirac equation:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(15)$$

or

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)q^a_\mu = 0. \quad -(16)$$

It is seen that the most fundamental elements of ϕ are V^a and V^μ, and that the generally covariant Dirac equation is:

$$(\Box + kT)\phi = 0 \quad -(17)$$

where:

$$\phi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} = \begin{bmatrix} q^R_1 \\ q^R_2 \\ q^L_1 \\ q^L_2 \end{bmatrix}. \quad -(18)$$

The tetrad is:

$$q^a_\mu = \begin{bmatrix} (\phi^R)^T \\ (\phi^L)^T \end{bmatrix} \quad -(19)$$

and the Dirac spinor is

$$\phi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad -(20)$$

4)

From eqs. (5) and (6):

$$q_\mu^a = (\phi^R)^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\phi^L)^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad -(21)$$

i.e.:

$$\begin{bmatrix} q_1^R & q_2^R \\ q_1^L & q_2^L \end{bmatrix} = \begin{bmatrix} q_1^R & q_2^R \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} q_1^L & q_2^L \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} q_1^R & q_2^R \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ q_1^L & q_2^L \end{bmatrix} \quad -(22)$$

Eq. (21) shows that the tetrad can be analyzed as the sum of two Pauli spinors, expressed in transposed form as row vectors. Therefore the fundamental Pauli spinors of Cartan geometry are made up of elements of column vector V^a and the chirality spin column vector V^μ.

During the course of this analysis have emerged the state spinors:

$$\xi_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \xi_2 = \begin{bmatrix}0\\1\end{bmatrix} \quad -(23)$$

which may be interpreted to mean ~~that the~~ fermion may be found in state $\underline{1}$ or 2. These are interconvertible by parity inversion:

$$\hat{P}(\xi_1) = \xi_2 \quad -(24)$$

and may therefore be considered as fundamental states of chirality or handedness.

The analysis has been carried out for the fermion field, with SU(2) symmetry. The SU(2) group is the unitary unimodular group of matrices such as:

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}. \quad -(25)$$

The hermitian transpose of S is:

$$S^+ = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix} \quad -(26)$$

and so:

$$SS^{\dagger} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(27)$$

and $\det S = 1$. $\quad -(28)$

Therefore this matrix has the required $SU(2)$ symmetry. It is a unitary matrix with a unit determinant. Note the similarity with the result:

$$\frac{1}{\sqrt{2}}(1-i)\frac{1}{\sqrt{2}}(1+i) = 1 \quad -(29)$$

which is the fundamental origin of the chirality vector in 7o(1):

$$\bar{V}^a = \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i \\ 1+i \end{bmatrix}. \quad -(30)$$

Eq. (30) can be further decomposed into:

$$\bar{V}^a = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\boxed{\bar{V}^a = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - i(\xi_1 - \xi_2)\right)} \quad -(31)$$

Therefore the chirality vector V^a is made up of a real part:

$$\mathrm{Re}\, V^a = \frac{1}{\sqrt{2}} \underline{1} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(32)$$

and an imaginary part:

$$\mathrm{Im}\, V^a = \frac{1}{\sqrt{2}} \left(\xi_2 - \xi_1 \right). \quad -(33)$$

These can be regarded as the fundamental geometrical elements of chirality, or handedness.

The unitary unimodular matrix S is made up of two Pauli matrices:

$$S = \frac{1}{\sqrt{2}} \left(\sigma_0 - i \sigma_1 \right) \quad -(34)$$

where:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad -(35)$$

The other two Pauli matrices are:

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad -(36)$$

The related matrix:

$$\xi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad -(37)$$

has the orthogonality property:

$$\xi \xi^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad -(38)$$

8) Thus:

$$S = \frac{1}{\sqrt{2}} \left(\underline{1} + i \left(\xi_2 - \xi_1 \right) \right) \quad - (39)$$

is unitary and unimodular, while:

$$\xi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \xi_2 - \xi_1 \\ 1 & \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\underline{1} \; \xi_2 - \xi_1 \right] \quad - (40)$$

is orthogonal.

Finally in this section we define the SU(2) matrix:

$$S = \begin{bmatrix} q_X^{(1)} & q_Y^{(1)} \\ -i q_X^{(1)} & -i q_Y^{(1)} \end{bmatrix} e^{-i\phi} \quad - (41)$$

where: $q_X^{(1)} = \frac{1}{\sqrt{2}} e^{i\phi}, \quad q_Y^{(1)} = -\frac{i}{\sqrt{2}} e^{i\phi} \quad - (42)$

and: $\underline{A}^{(1)} = A^{(0)} \underline{q}^{(1)} = A^{(0)} \left(q_X^{(1)} \underline{i} + q_Y^{(1)} \underline{j} \right) \quad - (43)$

This gives an SU(2) representation of the electromagnetic field. In the next section O(4), the gravitational field will be worked into these calculations.

i) PS

If we make the further decomposition:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad -(44)$$

$$= \xi_1 + \xi_2$$

it is found that the chirality vector is:

$$\boxed{\bar{V}^a = \frac{1}{\sqrt{2}}\left(\xi_1 + \xi_2 - i\left(\xi_1 - \xi_2\right)\right)} \quad -(45)$$

and the spin vector is:

$$\boxed{\bar{V}^\mu = \left(\xi_1 + \xi_2\right)e^{-i\phi}} \quad -(46)$$

so the fermion field (tetrad) in eq. (1) is defined entirely by:

$$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad -(47)$$

and the phase factor $e^{-i\phi}$. We define ξ_1 and ξ_2 to be the chiral elements of the field.