71(3): Frame Invariance of the Tetrad Postulate.

To Prove

$$D_\nu q^a_\mu = (D_\nu q^a_\mu)' = 0 \quad -(1)$$

under the general coordinate transformation.

Firstly some basic definitions of frame transformation are given with respect to the numbering given by Carroll:

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \quad -((2.11))$$

which is the frame transformation of the partial derivative.

A vector transforms as:

$$V^{\mu'} = \left(\frac{\partial x^{\mu'}}{\partial x^\mu} \right) V^\mu \quad -((2.13))$$

The Lorentz transform is the special case defined by:

$$x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu \quad -(2)$$

The symmetric metric transforms as:

$$g_{\mu'\nu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \right) \left(\frac{\partial x^\nu}{\partial x^{\nu'}} \right) g_{\mu\nu} \quad -(2.35)$$

The covariant derivative is defined as:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad -((3.1))$$

and is defined as a tensor, so that:

$$D_{\mu'} V^{\nu'} := \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right)\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) D_{\mu} V^{\nu} \quad -(3.2)$$

which mean that the Christoffel connection is not a tensor because it does not retain its form under the general coordinate transformation.

The general mixed index tensor transforms as:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_{a} \frac{\partial x^{\mu'}}{\partial x^{\mu}} \Lambda_{b'}{}^{b} \frac{\partial x^{\nu}}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu} \quad -(3)$$

as defined by Carroll in his chapter 3. Therefore the tetrad transforms as:

$$\boxed{q^{a'}_{\mu'} = \Lambda^{a'}{}_{a} \frac{\partial x^{\mu}}{\partial x^{\mu'}} q^{a}_{\mu}} \quad -(4)$$

There is one Lorentz transform $\Lambda^{a'}{}_{a}$ because the tangent spacetime is a Minkowski spacetime, and one general coordinate transform because the base manifold is a spacetime with curvature and torsion.

Therefore:

$$\boxed{\begin{aligned} D_{\nu'} q^{a'}_{\mu'} &= \left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}\right) \Lambda^{a'}{}_{a} \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) D_{\nu} q^{a}_{\mu} \\ &= 0 \end{aligned}} \quad -(5)$$

3) because:

$$D_\nu q^a_\mu = 0, \quad -(6)$$

Q.E.D.

Therefore the tetrad postulate is an invariant of Cartan geometry under the transformation from one frame of reference into any other frame of reference.

This means that:

$$\Box q^a_\mu = R q^a_\mu \quad -(7)$$

and

$$\Box' q^{a'}_{\mu'} = R' q^{a'}_{\mu'}. \quad -(8)$$

This means that the ECE Lemma is <u>covariant</u> under the transformation from one frame to another.

However, R is a <u>scalar</u> curvature, so:

$$R = R' \quad -(9)$$

and

$$\Box' q^{a'}_{\mu'} = R q^{a'}_{\mu'}. \quad -(10)$$

Also,

$$R = -kT \quad -(11)$$

in any frame of reference.

4)

So the ECE wave equation transforms as:

$$\left(\square' + kT\right) q^{a\,\prime}_{\mu'} = 0. \quad -(12)$$

For electrodynamics:

$$\left(\square + kT\right) A^{a}_{\mu} = 0 \quad -(13)$$

$$\downarrow$$

$$\left(\square' + kT\right) A^{a}_{\mu'} = 0. \quad -(14)$$

For fermion fields:

$$\left(\square + kT\right) \psi = 0 \quad -(15)$$

$$\downarrow$$

$$\left(\square' + kT\right) \psi' = 0. \quad -(16)$$

For gravitational fields:

$$\left(\square + kT\right) q^{a}_{\mu} = 0 \quad -(17)$$

$$\downarrow$$

$$\left(\square' + kT\right) q^{a\,\prime}_{\mu'} = 0 \quad -(18)$$

<u>Free Field Limit</u>

This is defined as:

$$kT \rightarrow \left(\frac{mc}{\hbar}\right)^2 \quad -(19)$$

5)

So the quantum of action is introduced by this limit:

$$\boxed{\hbar := \frac{mc}{(kT)^{1/2}}} \quad -(20)$$

The Proca equation, for example, is defined in this free field limit by:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) A^a_\mu = 0 \quad -(21)$$

$$\downarrow$$

$$\left(\Box' + \left(\frac{mc}{\hbar}\right)^2\right) A^{a'}_{\mu'} = 0 \quad -(22)$$

where m is the photon mass, whose existence has been verified to 1 : 100,000 by NASA Cassini. It is seen that the photon mass is well defined by the Proca type equation (21) and (22) in any frame of reference. The free field limit in this case means that the photon is a free photon uninfluenced by any other type of field.

Frame Covariance of the Field Equations of Electrodynamics

The field equations are rigorously frame

6)

covariant as follows (shorthand notation):

$$F = D \wedge A = d \wedge A + \omega \wedge A \quad -(23)$$

$$\downarrow$$

$$F' = (D \wedge A)' = (d \wedge A)' + (\omega \wedge A)' \quad -(24)$$

$$d \wedge F = \mu_0 j \quad -(25)$$

$$\downarrow$$

$$(d \wedge F)' = \mu_0 j' \quad -(26)$$

$$d \wedge \tilde{F} = \mu_0 J \quad -(27)$$

$$\downarrow$$

$$(d \wedge \tilde{F})' = \mu_0 J'. \quad -(28)$$

where:

$$j = \frac{A^{(0)}}{\mu_0} \left(R \wedge q - \omega \wedge T \right) \quad -(29)$$

$$\downarrow$$

$$j' = \frac{A^{(0)}}{\mu_0} \left(R \wedge q - \omega \wedge T \right)' \quad -(30)$$

This means that Cartan geometry is rigorously frame covariant.

It is seen that the most basic property is the frame invariance of the tetrad postulate. This replaces the gauge principle in ECE theory. The tetrad postulate is always true in any frame of reference. In electrodynamics:

$$\boxed{D_\nu A_\mu^a = (D_\nu A_\mu^a)' = 0} \quad -(31)$$

and this is a basic principle of generally covariant electrodynamics.

Some Remarks on Gauge Theory

As described by Carroll in chapter three, gauge theory is superfluous, in that it introduces an abstract "internal" vector space. This has no relation to the base manifold and has to be defined by "an independent addition to the manifold" (Carroll p. 94). The fibre bundle is the union of the base manifold and the internal vector space. The gauge transform is the transform of a field defined in the fibre bundle. The gauge principle is that physical quantities are invariant under a gauge transform in the fibre bundle. Gauge theories are invariant under gauge transformation

8) Without going into any details of gauge theory it is seen that there is an additional postulate of the internal vector space which does not exist in ECE theory, where everything is defined geometrically. The basic definition of ECE is the basic definition of Cartan geometry:

$$V^a = q^a_\mu V^\mu \quad -(32)$$

and the tetrad postulate follows from this. The interaction between fields in ECE theory is defined by <u>the Bianchi identity</u>:

$$D \wedge T = d \wedge T + \omega \wedge T := R \wedge q \quad -(33)$$

and not by the gauge transform. Eq. (33) defines the interaction between "translation" R and "rotation" T.

<u>PS On the d'Alembertian</u>

This is also <u>frame invariant</u> because it is defined by:

$$\Box = \partial^\mu \partial_\mu = \Box' \quad -(34)$$

So:

$$\boxed{\Box q^a_\mu = R q^a_\mu \rightarrow \Box q^{a\,\prime}_\mu = R q^{a\,\prime}_\mu} \quad -(35)$$

9)

Therefore the ECE wave equation transforms as:

$$\boxed{\begin{gathered}(\square + kT)\, q^a_\mu = 0 \\ \downarrow \\ (\square + kT)\, q^{a\prime}_{\mu}{}^{\prime} = 0\end{gathered}} \quad \begin{matrix} -(36) \\ \\ -(37) \end{matrix}$$

- under the coordinate transformation from one frame of reference to any other, moving arbitrarily with respect to the first.