1)

# 78(5): Development of Cartan Geometry in SU(n).

## SU(2)

The basis set in 3-D is defined by the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(1)$$

so:

$$\underline{\sigma} \cdot \underline{r} = \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix}. \quad -(2)$$

The complete potential field is defined by the following tetrad definition:

$$A_\mu = A_\mu^a \frac{\sigma_a}{2}. \quad -(3)$$

and the electromagnetic field is:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\epsilon_{abc} A_\mu^b A_\nu^c. \quad -(4)$$

So for example:

$$F_{\mu\nu}^1 = \partial_\mu A_\nu^1 - \partial_\nu A_\mu^1 + g\,\epsilon_{1bc} A_\mu^b A_\nu^c$$

$$= \partial_\mu A_\nu^1 - \partial_\nu A_\mu^1 + g\left(A_\mu^2 A_\nu^3 - A_\mu^3 A_\nu^2\right) \quad -(5)$$

This is a particular case of:

2)

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b \quad -(6)$$

when there is no effect of gravitation. In this case:

$$\omega^a{}_b = -\frac{1}{2} \kappa \epsilon^a{}_{bc} q^c \quad -(7)$$

where:

$$A^a = A^{(0)} q^a, \quad -(8)$$

and the homogeneous current vanishes:

$$j^a = \frac{A^{(0)}}{\mu_0} \left( R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right) \quad -(9)$$

$$= 0.$$

So in the absence of gravitation:

$$d \wedge F^a = 0 \quad -(10)$$

$$d \wedge \tilde{F}^a = \mu_0 J^a. \quad -(11)$$

In the complex circular basis:

$$\underline{B}^{(1)*} = \underline{\nabla} \times \underline{A}^{(1)*} - ig \underline{A}^{(2)} \times \underline{A}^{(3)} \quad -(12)$$

$$\underline{B}^{(2)*} = \underline{\nabla} \times \underline{A}^{(2)*} - ig \underline{A}^{(3)} \times \underline{A}^{(1)} \quad -(13)$$

$$\underline{B}^{(3)*} = \underline{\nabla} \times \underline{A}^{(3)*} - ig \underline{A}^{(1)} \times \underline{A}^{(2)} \quad -(14)$$

3) If $\underline{A}^{(3)}$ is Z directed then:

$$\underline{B}^{(3)*} = \underline{B}^{(3)} = -ig\,\underline{A}^{(1)} \times \underline{A}^{(2)} \quad -(15)$$

which is the fundamental spin field. Using the SU(2) basis gives the extra information:

$$\underline{\sigma} \cdot \underline{B}^{(3)} = \begin{bmatrix} B_Z & 0 \\ 0 & -B_Z \end{bmatrix}$$

$$= -ig\underline{\sigma} \cdot \underline{A}^{(1)} \times \underline{A}^{(2)}. \quad -(16)$$

which produces radiatively induced fermion resonance (RFR). Gravitation changes this result by changing the conjugate product $\underline{A}^{(1)} \times \underline{A}^{(2)}$.

## Rule for Developing Electrodynamics in SU(2)

From eq. (4), electrodynamics can be developed in SU(2) using the same Cartan geometry as in O(3). However, extra information can be found at the end of the calculation by using the Pauli matrices as for example in eq. (16). This rule can also be used for SU(n), $n \geq 2$.

4)

## SU(3)

This is the group that governs gluons in the quark gluon model. From the foregoing it is clear that electrodynamics can also be developed in SU(3), or in any SU(n) symmetry. In SU(3) the complete potential field is:

$$A_\mu = A_\mu^a \lambda_a / 2 \quad — (17)$$

where $\lambda_a$ are the SU(3) matrices. The electromagnetic field is:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c \quad - (18)$$

where $f_{abc}$ is the SU(3) structure factor. In eq. (4) the tangent space of Cartan geometry is represented by a SU(2) <u>representation space</u>, and in eq. (18) by a SU(3) representation space. The latter was introduced by Cartan himself in 1913. Eq. (18) is again a special case of the more general eq. (6). <u>This fact illustrates the generality of Cartan geometry.</u>

5) Eq. (18) is the special case:

$$\omega^a_{\mu b} \wedge q^b_\nu = g f_{abc} A^b_\mu A^c_\nu \quad -(19)$$

and Eq. (4) is the special case:

$$\omega^a_{\mu b} \wedge q^b_\nu = g \epsilon_{abc} A^b_\mu A^c_\nu. \quad -(20)$$

This process can be continued in any $SU(n)$ symmetry, $n = 2, 3, 4, \ldots\ldots$ In general:

$$\omega^a_{\mu b} \wedge q^b_\nu = g \zeta_{abc} A^b_\mu A^c_\nu \quad -(21)$$

where $\zeta_{abc}$ is the general structure factor of the group $SU(n)$.

<u>The Effect of Gravitation in General</u>

It is seen that in general, <u>the effect of gravitation is to break the $SU(n)$ symmetry</u>. In the absence of gravitation the general symmetry is:

$$\boxed{\omega^a{}_b = -\frac{1}{2} \kappa \zeta^a{}_{bc} q^c} \quad -(22)$$

but this is no longer true in the presence of gravitation.

Standard Model

In this case, consideration is restricted to the $u(1) = o(2)$ group, so:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad -(23)$$

which in differential form notation is:

$$F = d \wedge A \quad -(24)$$

So in the standard model there is no self-consistent method of describing the observable inverse Faraday effect. In ECE, and for any representation space, the inverse Faraday effect is the result of the spin connection term in eq. (6), i.e. we observe spinning spacetime in the inverse Faraday effect.

Extra information about classical electrodynamics may be found by the method of eq. (16) in all $SU(n)$, not only in $SU(2)$. In $SU(3)$ for example we multiply $\underline{B}^{(3)}$ by $3 \times 3$ matrices to describe the gluon / the $SU(3)$ e/m interaction.