1) $\nabla^2 \psi = -\frac{4mc\pi}{\hbar} x \psi$; — (1)

$x = \frac{1}{r} - \frac{1}{r + r(vac)}$

LAMB SHIFT IN H

AVERAGE $\langle \psi_n(2p) \rangle$

$\psi_n(1s) = \exp\left(-\frac{r}{a}\right)$

$\psi_n(2s) = \left(2 - \frac{r}{a}\right) \exp\left(-\frac{r}{2a}\right)$

$\psi_n(2p_z) = \frac{r}{a} \cos\theta \exp\left(-\frac{r}{2a}\right)$

$\psi_n(2p_x) = -\frac{r}{a} \sin\theta \, e^{i\phi} \exp\left(-\frac{r}{2a}\right)$

$\psi_n(2p_y) = \frac{r}{a} \sin\theta \, e^{-i\phi} \exp\left(-\frac{r}{2a}\right)$

$\psi_n^2(2p_z) = \left(\frac{r}{a}\right)^2 \cos^2\theta \exp\left(-\frac{r}{a}\right)$

$\psi_n(2p_x)\psi_n(2p_y) = -\left(\frac{r}{a}\right)^2 \sin^2\theta \exp\left(-\frac{r}{a}\right)$

$\psi_n^2(2p_z) - \psi_n(2p_x)\psi_n(2p_y) = \left(\frac{r}{a}\right)^2 \exp\left(-\frac{r}{a}\right)$

$\left(\psi_n^2(2p_z - \psi_n(2p_x)\psi_n(2p_y))\right)^{1/2} = \frac{r}{a} \exp\left(-\frac{r}{2a}\right)$

$:= \langle \psi_n(2p) \rangle$

$$\boxed{\psi = \psi_n(2s) - \langle \psi_n(2p) \rangle = 2\left(1 - \frac{r}{a}\right) \exp\left(-\frac{r}{2a}\right)} \quad \text{— (2)}$$

Now use eq (2) in eq (1):

2)

$$\nabla^2 \varphi = \frac{2}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial r^2}$$

$$\frac{\partial \varphi}{\partial r} = -\frac{2}{a}\exp\left(-\frac{r}{2a}\right) - \frac{2}{2a}\left(1-\frac{r}{a}\right)\exp\left(-\frac{r}{2a}\right)$$

$$= -\frac{e^{-r/2a}}{a}\left(3 - \frac{r}{a}\right) = -\frac{1}{a}\left(3-\frac{r}{a}\right)e^{-r/(2a)}$$

$$\frac{\partial^2 \varphi}{\partial r^2} = \frac{1}{a^2}\exp\left(-\frac{r}{2a}\right) + \frac{1}{2a^2}\left(3-\frac{r}{a}\right)\exp\left(-\frac{r}{2a}\right)$$

$$= \frac{1}{a^2}\left(\frac{5}{2} - \frac{r}{2a}\right)\exp\left(-\frac{r}{2a}\right)$$

$$= \frac{1}{2a^2}\left(5-\frac{r}{a}\right)\exp\left(-\frac{r}{2a}\right)$$

$$\nabla^2 \varphi = \left(\frac{1}{2a^2}\left(5-\frac{r}{a}\right) - \frac{2}{ar}\left(3-\frac{r}{a}\right)\right)\exp\left(-\frac{r}{2a}\right)$$

$$= -\frac{4mc\pi}{\hbar}x \cdot 2\left(1-\frac{r}{a}\right)\exp\left(-\frac{r}{2a}\right)$$

$$\boxed{x = \frac{\hbar}{8\pi mc}\,\frac{\left(\frac{2}{ar}\left(3-\frac{r}{a}\right) - \frac{1}{2a^2}\left(5-\frac{r}{a}\right)\right)}{\left(1-\frac{r}{a}\right)}}$$

3)

Therefore:

$$\boxed{x = \frac{1}{r} - \frac{1}{r + r(vac)} = \frac{1}{8\pi mca}\,\hbar \left(\frac{6}{r} - \frac{7}{a} + \frac{r}{2a^2}\right) \Big/ \left(1 - \frac{r}{a}\right.}$$

i.e. $\Delta \bar{V} = -\alpha x \quad cm^{-1}$

$$\boxed{\Delta E = -\frac{\alpha x r}{2n^2 a}} \quad cm^{-1}$$

Lamb shift

Alternative Calculation

Use:

$$\nabla^2 \phi_n(2s) = -\frac{4mc\pi}{\hbar} x_1 \phi_n(2s) \quad -(3)$$

$$\nabla^2 \langle \phi_n(2p) \rangle = -\frac{4mc\pi}{\hbar} x_2 \langle \phi_n(2p) \rangle \quad -(4)$$

From eq. (3):

$$\boxed{x_1 = \frac{\hbar}{4\pi mca}\left(\frac{2}{r} - \frac{1}{a}\right)} \quad -(5)$$

In eq. (4) consider:

$$\nabla^2 \left(\frac{r}{a} \exp\left(-\frac{r}{2a}\right)\right) = \nabla^2 \phi \quad -(6)$$

4)

$$\frac{\partial \phi}{\partial r} = \exp\left(-\frac{r}{2a}\right)\left(\frac{1}{a} - \frac{r}{2a^2}\right)$$

$$\frac{\partial \phi}{\partial r} = \frac{1}{a}\left(1 - \frac{r}{2a}\right)\exp\left(-\frac{r}{2a}\right)$$

$$\frac{\partial^2 \phi}{\partial r^2} = \exp\left(-\frac{r}{2a}\right)\left(-\frac{1}{2a^2} - \frac{1}{2a^2}\left(1 - \frac{r}{2a}\right)\right)$$

$$= \exp\left(-\frac{r}{2a}\right)\left(-\frac{1}{a^2} + \frac{r}{4a^3}\right)$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{1}{a^2}\left(\frac{r}{4a} - 1\right)\exp\left(-\frac{r}{2a}\right)$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r}$$

$$\nabla^2 \phi = \left(\frac{1}{a^2}\left(\frac{r}{4a} - 1\right) + \frac{2}{ar}\left(1 - \frac{r}{2a}\right)\right)e^{-r/(2a)}$$

$$= -\frac{4mc\pi}{h} x_2 \frac{r}{a} e^{-r/(2a)}$$

$$x_2 = \frac{h}{4\pi mc}\frac{a}{r}\left(\frac{1}{a^2}\left(1 - \frac{r}{4a}\right) + \frac{2}{ar}\left(\frac{r}{2a} - 1\right)\right)$$

$$= \frac{h}{4\pi mc}\frac{a}{r}\left(\frac{2}{a^2} - \frac{r}{4a^3} - \frac{2}{ar}\right)$$

5)

$$x_2 = \frac{\hbar}{4\pi m c a}\left(\frac{2}{r} - \frac{1}{4a} - \frac{2a}{r^2}\right) \qquad -(7)$$

$$x_1 - x_2 = \frac{\hbar}{4\pi m c a}\left(\frac{2a}{r^2} - \frac{3}{4a}\right)$$

$$\boxed{x_1 - x_2 = \frac{\hbar}{2\pi m c}\,\frac{1}{r^2} - \frac{3\hbar}{16\pi m c a^2}} \qquad -(8)$$

$$\Delta V = -\alpha\,(x_1 - x_2)$$

$$\Delta E = -\frac{r}{\lambda^2 a}\,\Delta V$$ — Lamb Shift