## (3): Vacuum Effect on the Dirac Equation of H.

It has been shown in note 86(2) that the Dirac equation of a free electron:

$$i\gamma^\mu \partial_\mu \psi_0 = \frac{mc}{\hbar}\psi_0 \quad -(1)$$

can be written as:

$$i\left(\frac{\partial\psi}{\partial t} + c\frac{\partial\psi^*}{\partial Z}\right) = \frac{mc^2}{\hbar}\psi \quad -(2)$$

if the electron is moving in Z. Here:

$$\psi = \psi_1^L + \psi_2^L + \psi_1^R + \psi_2^R = \exp\left(-\frac{imc^2}{2\hbar}\left(t - \frac{Z}{c}\right)\right) \quad -(3)$$

$$\psi^* = \psi_2^L - \psi_1^L + \psi_1^R - \psi_2^R = \exp\left(\frac{imc^2}{2\hbar}\left(t - \frac{Z}{c}\right)\right). \quad -(4)$$

This method suggests that a simple way may be found for solving the Dirac equation of atomic H:

$$i\gamma^\mu \partial_\mu \psi_H = \left(\frac{mc}{\hbar} + \frac{\alpha}{r}\right)\psi_H \quad -(5)$$

and then incorporating the Lamb shift as:

$$\boxed{i\gamma^\mu \partial_\mu \left(1 + \frac{\alpha}{4\pi}\right)\psi_1 = \left(\frac{mc}{\hbar} + \frac{\alpha}{r}\right)\psi_1} \quad -(6)$$

The wavefunction or spinor of eq. (6) is different from $\psi_H$ because it is affected by the vacuum. From eq. (5):

$$E_L(2S_{1/2}) = E_L(2P_{1/2}) \quad -(7)$$

but in eq. (6):

$$E_L(2S_{1/2}) > E_L(2P_{1/2}) \quad -(8)$$

producing the Lamb shift.

A correctly relativistic treatment of the Lamb shift in the H atom is ~~needed~~ to produce ... In ECE theory this is given self-consistently by eq. (6). In QED a very complicated explanation is attempted using virtual photons, virtual electron-positron pairs, and adjustable parameters of the path integral formalism.

Following the method of paper 85, eqn. (6) is expressed in terms of $r(\text{vac})$:

$$\boxed{i\gamma^\mu \partial_\mu \psi_1 = \left( \frac{mc}{\hbar} + \frac{\alpha}{r} - \frac{\alpha}{r(\text{vac})} \right) \psi_1} \quad -(9)$$

using:

$$\frac{i\alpha}{4\pi}\gamma^\mu \partial_\mu \phi_1 = \frac{\alpha}{r(vac)}\phi_1 \quad \text{---} (10)$$

i.e

$$i\gamma^\mu \partial_\mu \phi_1 = \frac{4\pi}{r(vac)}\phi_1 . \quad \text{---} (11)$$

It is known experimentally that the Lamb shift changes the total energy of the $2S_{1/2} = 2P_{1/2}$ orbital by about two parts in ten million. Therefore, in eq. (11):

$$\phi_1 \sim \phi_H \quad \text{---} (12)$$

and this produces:

$$E_n(2S_{1/2}) > E_n(2P_{1/2}) . \quad \text{---} (13)$$

by about 0.03 $cm^{-1}$. So:

$$\boxed{i\gamma^\mu \partial_\mu \phi_H = \frac{4\pi}{r(vac)}\phi_H} \quad \text{---} (14)$$

to a very good approximation.

Eq (14) must be solved using the known Dirac orbitals of the H atom. The $r(vac)$ is found from the experimentally measured Lamb shift. In paper 85 this was done in the non-relativistic approximation using the Schrödinger orbitals of the H atom. The result of that calculation produced a finite electron radius of the same order of magnitude as r(vac).

<u>H Radial Functions of the Dirac Equation</u>

These are derived by E. Merzbacher, "Quantum Mechanics" (Wiley, 1970, 2nd. ed.), pp. 603 ff. The Hamiltonian is

$$\hat{H} = c\,\underline{\alpha} \cdot \frac{\underline{\hbar}}{i} \underline{\nabla} + \beta \mu c^2 - e\phi_0(r) \quad -(15)$$

in his notation. The total angular momentum operator is:

$$\underline{J} = \underline{r} \times \frac{\hbar}{i} \underline{\nabla} + \frac{\hbar}{2} \underline{\Sigma} \quad -(16)$$

The spinor is expressed as:

$$\psi = \begin{bmatrix} \phi \\ \chi \end{bmatrix} \quad -(17)$$

so:

$$(E - \mu c^2 + e\phi_0)\,\phi - \frac{\hbar}{i} c\,\underline{\sigma} \cdot \underline{\nabla} \chi = 0 \quad -(18)$$

$$(E + \mu c^2 + e\phi_0)\,\chi - \frac{\hbar}{i} c\,\underline{\sigma} \cdot \underline{\nabla} \phi = 0 \quad -(19)$$

The operators $J_z$ and:

$$J^2 = L^2 + \hbar\, \underline{L} \cdot \underline{\Sigma} + \frac{3}{4}\hbar^2$$

) must satisfy:

$$\left(L_z + \frac{\hbar}{2}\sigma_z\right)\begin{bmatrix}\phi\\ \chi\end{bmatrix} = m\hbar\begin{bmatrix}\phi\\ \chi\end{bmatrix} \quad (21)$$

and

$$\left(L^2 + \hbar\underline{L}\cdot\underline{\sigma} + \frac{3}{4}\hbar^2\right)\begin{bmatrix}\phi\\ \chi\end{bmatrix} = j(j+1)\hbar^2\begin{bmatrix}\phi\\ \chi\end{bmatrix} \quad (22)$$

For a given value of $j$ the spinors $\phi$ and $\chi$ are proportional to the functions $Y^{jm}_{j\mp(1/2)}$

where:

$$\int Y^{+} Y \, d\Omega = 1 \quad (23)$$

and:

$$\underline{\sigma}\cdot\underline{\hat{r}}\, Y^{jm}_{j\mp(1/2)} = -Y^{jm}_{j\pm(1/2)} \quad (24)$$

The solution of the Dirac equation for H is then:

$$\psi = \begin{bmatrix} F(r)\, Y^{jm}_{j-(1/2)} \\ -i f(r)\, Y^{jm}_{j+(1/2)} \end{bmatrix} \quad (25)$$

or

$$\psi = \begin{bmatrix} G(r)\, Y^{jm}_{j+(1/2)} \\ -i g(r)\, Y^{jm}_{j-(1/2)} \end{bmatrix} \quad (26)$$

6)

If we define:

$$\lambda = j + 1/2, \quad \frac{E}{\mu c^2} = \epsilon, \quad x = r/(\hbar/\mu c), \; e^2/(\hbar c) = \alpha,$$

then:

$$\left(\epsilon - 1 + \frac{\alpha}{x}\right) F - \left(\frac{d}{dx} + \frac{\lambda + 1}{x}\right) f = 0 \quad -(27)$$

$$\left(\epsilon + 1 + \frac{\alpha}{x}\right) f + \left(\frac{d}{dx} - \frac{\lambda - 1}{x}\right) F = 0 \quad -(28)$$

and

$$\left(\epsilon - 1 + \frac{\alpha}{x}\right) G - \left(\frac{d}{dx} - \frac{\lambda - 1}{x}\right) g = 0 \quad -(29)$$

$$\left(\epsilon + 1 + \frac{\alpha}{x}\right) g + \left(\frac{d}{dx} + \frac{\lambda + 1}{x}\right) G = 0 \quad -(30)$$

From these equations:

$$F = \exp\left(-(1-\epsilon^2)^{1/2} x\right) x^\gamma \sum_{\nu=0} a_\nu x^\nu \quad -(31)$$

$$f = \exp\left(-(1-\epsilon^2)^{1/2} x\right) x^\gamma \sum_{\rho=0} b_\rho x^\rho \quad -(32)$$

where:

$$\gamma = -1 + \left((j + 1/2)^2 - \alpha^2\right)^{1/2}$$

for $\nu = 0$ and $a_{-1} = b_{-1} = 0$, $\quad -(33)$

and:

7)

$$b_{n'} = \left( \frac{1-\epsilon}{1+\epsilon} \right)^{1/2} a_{n'}, \quad -(34)$$

with:

$$(1-\epsilon^2)^{1/2}(1+\gamma+n') = \alpha\epsilon. \quad -(35)$$

So the computer algebra problem is to find $r(vac)$ for eqns. (14) and (31) to (35). In the absence of the vacuum:

$$E_n = \mu c^2 \left( 1 + \frac{\alpha^2}{\left( \left( (j+1/2)^2 - \alpha^2 \right)^{1/2} + n' \right)} \right)^{-1/2}$$

$-(36)$

This is the structure formula of the H atom.

Here:

$$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots ; \quad n' = 0, 1, 2, \ldots$$

$-(37)$

and the principal quantum number of the non-relativistic H atom is:

$$n = j + 1/2 + n' \quad -(38)$$

8)

In general:

$$(\epsilon-1)a_{n-1}+\alpha a_n+(1-\epsilon^2)^{1/2}b_{n-1}-(\lambda+1+\gamma+n)b_n=0 \quad -(39)$$

and

$$(\epsilon+1)b_{n-1}+\alpha b_n-(1-\epsilon^2)^{1/2}a_{n-1}+(-\lambda+1+\gamma+n)a_n=0 \quad -(40)$$

For

$$n=n'+1 \quad -(41)$$

with $a_{n'+1}=b_{n'+1}=0$:

$$b_{n'}=\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/2}a_{n'} \quad -(42)$$

Now simultaneously eliminate $a_{n-1}$ and $b_{n-1}$ from eqs. (39) and (40) to obtain:

$$a_n\left(\alpha(1+\epsilon)^{1/2}+(\lambda-1-\gamma-n)(1-\epsilon)^{1/2}\right)=b_n\left(\alpha(1-\epsilon)^{1/2}+(\lambda+1+\gamma+n)(1+\epsilon)^{1/2}\right) \quad -(43)$$

Finally let $n=n'$ and compare with eq. (42) to obtain eq. (35)