88(a): Summary of the Bianchi Identities

Second Bianchi Identity

$$D\wedge R^a{}_b = D_\rho R^a{}_{b\mu\nu} + D_\mu R^a{}_{b\nu\rho} + D_\nu R^a{}_{b\rho\mu}$$

$$:= q^\sigma{}_b \left(D_\rho D_\sigma T^a{}_{\mu\nu} + D_\mu D_\sigma T^a{}_{\nu\rho} + D_\nu D_\sigma T^a{}_{\rho\mu} \right)$$

$$\boxed{D\wedge R^a{}_b := q^\sigma{}_b \, D\wedge \left(D_\sigma T^a\right).} \qquad -(1)$$

Possible solutions of this equation include the following.

a)

$$\left[\begin{array}{ll} D_\sigma T^a{}_{\mu\nu} = R^a{}_{\sigma\mu\nu} & -(2) \\ D_\sigma T^a{}_{\nu\rho} = R^a{}_{\sigma\nu\rho} & -(3) \\ D_\sigma T^a{}_{\rho\mu} = R^a{}_{\sigma\rho\mu} & -(4) \end{array}\right.$$

b)

$$\left[\begin{array}{ll} D_\rho R^a{}_{b\mu\nu} + D_\mu R^a{}_{b\nu\rho} + D_\nu R^a{}_{b\rho\mu} = 0, & -(5) \\ T^a{}_{\mu\nu} = T^a{}_{\nu\rho} = T^a{}_{\rho\mu} = 0 & -(6) \end{array}\right.$$

c)

$$\left[\begin{array}{ll} R^a{}_{b\mu\nu} = R^a{}_{b\nu\rho} = R^a{}_{b\rho\mu} = 0 & -(7) \\ D_\rho D_\sigma T^a{}_{\mu\nu} + D_\mu D_\sigma T^a{}_{\nu\rho} + D_\nu D_\sigma T^a{}_{\rho\mu} = 0 & -(8) \end{array}\right.$$

Standard cosmology is restricted to case (b) only, eqs. (2) - (4) define a new form of torsion, and show that the Riemann tensor can always be expressed as this new type of torsion.

2)

First Bianchi Identity

$$\boxed{R^a{}_b \wedge q^b := D \wedge T^a} \quad - (9)$$

where

$$T^a = D \wedge q^a \quad - (10)$$

The case considered by Einstein and Hilbert is:

$$\begin{cases} R^a{}_b \wedge q^b = 0 & - (11) \\ T^a = 0 & - (12) \\ D \wedge R^a{}_b = 0 & - (13) \end{cases}$$

where:

$$R^a{}_b = D \wedge \omega^a{}_b . \quad - (14)$$

In tensor notation, eq. (9) is:

$$D_\rho T^a{}_{\mu\nu} + D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} \quad - (15)$$

$$= R^a{}_{\rho\mu\nu} + R^a{}_{\mu\nu\rho} + R^a{}_{\nu\rho\mu}$$

from which it is again possible to define the new type of torsion in eqns. (2) - (4).