112(4): Derivation of the Newton Inverse Square Law.

The starting point is the hypothesis:

$$R^{a}{}_{\rho\mu\nu} = D_\rho S^{a}{}_{\mu\nu} \quad — (1)$$

and the Newton inverse square law will be derived from this in the weak field limit.

The right hand side of eq. (1) is the covariant derivative. In tensor notation, eq. (1) is:

$$R^{\kappa}{}_{\rho\mu\nu} = D_\rho S^{\kappa}{}_{\mu\nu} \quad — (2)$$

where

$$S^{\kappa}{}_{\mu\nu} = - S^{\kappa}{}_{\nu\mu}. \quad — (3)$$

The rule for the covariant derivative of a tensor of any rank is given by Carroll:

$$D_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l}$$
$$+ \Gamma^{\mu_2}{}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots - \Gamma^{\lambda}{}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l}$$
$$- \Gamma^{\lambda}{}_{\sigma\nu_2} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - \ldots \quad — (4)$$

For example:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda \quad — (5)$$

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^{\lambda}_{\nu\mu} V_\lambda \quad — (6)$$

$$D_\mu V^{\nu\rho} = \partial_\mu V^{\nu\rho} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda\rho} + \Gamma^{\rho}_{\mu\lambda} V^{\nu\lambda} \quad — (7)$$

Therefore:

2)

$$D_\sigma S^\mu{}_{\nu\rho} = \partial_\sigma S^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\sigma\lambda} S^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\sigma\nu} S^\mu{}_{\lambda\rho} - \Gamma^\lambda{}_{\sigma\rho} S^\mu{}_{\nu\lambda} \quad -(8)$$

By inspection we define:

$$\boxed{R^\mu{}_{\sigma\nu\rho} := D_\nu S^\mu_{\rho\sigma}} \quad -(9)$$

Here:

$$R^\mu{}_{\sigma\nu\rho} = \partial_\nu \Gamma^\mu_{\rho\sigma} - \partial_\rho \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma}, \quad -(10)$$

$$D_\nu S^\mu_{\rho\sigma} = \partial_\nu S^\mu_{\rho\sigma} + \Gamma^\mu_{\nu\lambda} S^\lambda_{\rho\sigma} - \Gamma^\lambda_{\nu\rho} S^\mu_{\lambda\sigma} - \Gamma^\lambda_{\nu\sigma} S^\mu_{\rho\lambda}. \quad -(11)$$

Particular Solutions

$$\Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} = \Gamma^\lambda_{\nu\sigma} S^\mu_{\rho\lambda} \quad -(12)$$

$$\Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma} = \Gamma^\mu_{\nu\lambda} S^\lambda_{\rho\sigma} \quad -(13)$$

$$\partial_\nu \Gamma^\mu_{\rho\sigma} = \partial_\nu S^\mu_{\rho\sigma} \quad -(14)$$

$$\partial_\rho \Gamma^\mu_{\nu\sigma} = \Gamma^\lambda_{\nu\rho} S^\mu_{\lambda\sigma} \quad -(15)$$

Solving eqns (12) to (15):

$$\boxed{\begin{aligned} &S^\lambda_{\rho\sigma} = \Gamma^\lambda_{\rho\sigma} && -(16) \\ \text{if:}\quad &\partial_\rho \Gamma^\mu_{\nu\sigma} = \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} && -(17) \end{aligned}}$$

3)

Eq. (16) identifies $S^{\lambda}_{\rho\sigma}$ as the gamma connection provided the latter is constrained by eq. (17).

Einstein Hilbert Limit

In this limit:

$$\Gamma^{\lambda}_{\rho\sigma} = \Gamma^{\lambda}_{\sigma\rho}, \quad -(18)$$

which is the Christoffel symbol.

Newtonian Limit

This is described in Carroll, chapter 4, where the Einstein Hilbert field equation is reduced to the Poisson equation:

$$\nabla^2 \Phi = 4\pi G\rho \quad -(19)$$

which is equivalent to the Newton inverse square law. Here ρ is the mass density, G is Newton's constant and Φ is the gravitational potential. The EH field equation is:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} \quad -(20)$$

which can be contracted to:

$$R = -kT \quad -(21)$$

so:

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right). \quad -(22)$$

The Newtonian limit is the weak field, time-independent, slowly moving particles limit.

4) In this limit $T_{\mu\nu}$ is dominated by the rest energy, so:

$$\rho = T_{00} = \frac{m}{V} \quad -(23)$$

In this limit the relevant metric components are:

$$g_{00} = -1 + h_{00} \quad -(24)$$

$$g^{00} = -1 - h_{00} \quad -(25)$$

so:

$$T = g^{00} T_{00} = -T_{00}. \quad -(26)$$

Therefore from eq. (22):

$$R_{00} = \frac{1}{2} k T_{00} \quad -(27)$$

$$= R^{\lambda}{}_{0\lambda 0}.$$

Since

$$R^{0}{}_{000} = 0 \quad -(28)$$

it is necessary to consider:

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0} \quad -(29)$$

$$\boxed{R^{i}{}_{0j0} \sim \partial_j \Gamma^{i}_{00}} \quad -(29a)$$

in the weak field limit, because terms to second order are neglected and the field is slowly varying in time, so:

$$\partial_0 \Gamma^{i}_{j0} \sim 0. \quad -(30)$$

The Christoffel symbol is ~~ex~~ defined in terms of the symmetric metric, so:

$$R_{00} = R^{i}{}_{0i0} = \partial_i \left(\frac{1}{2} g^{i\lambda} \left(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00} \right) \right) \quad -(31)$$

The time derivatives are neglected, so:

$$R_{00} = R^{i}{}_{0i0} = -\frac{1}{2} \partial_i g^{i\lambda} \partial_\lambda g_{00}$$

$$= -\frac{1}{2} \partial_i g^{i\lambda} \partial_\lambda \left(-1 + h_{00} \right)$$

$$= -\frac{1}{2} \eta^{ij} \partial_i \partial_j h_{00}$$

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00} = \frac{1}{2} k T_{00} \quad -(32)$$

Therefore:

$$\nabla^2 h_{00} = -k\rho \quad -(33)$$

This is eq. (19) if:

$$k = \frac{8\pi G}{c^2}, \quad h_{00} = -\frac{2\Phi}{c^2} \quad -(34)$$

Derivation from Eq. (9)

Using the above standard derivation as a guideline, it is shown that the derivation of the Newton inverse square law from eq. (9) is simpler and more straightforward.

6)

In the weak field limit, term to second order in eq (9) are neglected, so:

$$\boxed{D_\sigma S^{\mu}{}_{\nu\rho} \sim \partial_\sigma S^{\mu}{}_{\nu\rho} = R^{\mu}{}_{\sigma\nu\rho}} \quad -(35)$$

Thus:

$$\partial_j S^{i}{}_{00} = R^{i}{}_{0j0} \quad -(36)$$

and:

$$\boxed{\partial_i S^{i}{}_{00} = R_{00} = R^{i}{}_{0i0} = k T_{00} = k\rho} \quad -(36)$$

Now compare:

$$\partial_i S^{i}{}_{00} = k\rho \quad -(37)$$

and

$$\nabla^2 \Phi = 4\pi G\rho \quad -(38)$$

using

$$\nabla^2 = -\partial_i \partial^i, \quad k = \frac{8\pi G}{c^2} \quad -(39)$$

Eq. (38) is:

$$-\partial_i \partial^i \Phi = 4\pi G\rho \quad -(40)$$

i.e.

$$-\partial_i\left(\partial^i\left(\frac{2\Phi}{c^2}\right)\right) = \frac{8\pi G}{c^2}\rho \quad -(41)$$

so

$$S_{00}{}^{i} = -\frac{2}{} \partial^i \Phi \quad -(42)$$

7) or in vector notation:

$$\boxed{\underline{S}_{00} = -\frac{2}{c^2}\underline{\nabla}\,\underline{\Phi}} \quad -(43)$$

However, we know that:

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi} \quad -(43)$$

(Marion and Thornton, page 159). So:

$$\boxed{\underline{S}_{00} = \frac{2}{c^2}\underline{g}} \quad -(44)$$

where $\underline{g}$ is the acceleration due to gravity.

Units

$$\underline{S}_{00} = m^{-1}, \quad \underline{g} = m s^{-2}, \quad c^2 = m^2 s^{-2}$$

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Summary

1) The Newton inverse square law is obtained straightforwardly from eq. (9), and the acceleration due to gravity is:

$$\boxed{\underline{g} = \frac{1}{2}c^2\underline{S}_{00}} \quad -(45)$$

2) If the Riemann tensor is defined by eq. (9), the Γ symbols are constrained by eqs. (16) and (17); under this constraint, the Newtonian acceleration due to gravity is defined by eq. (45).

3) The field equation of gravitation in general is given by:

$$\boxed{D_\nu S^{\mu}{}_{\rho\sigma} = R^{\mu}{}_{\sigma\nu\rho}} \quad (46)$$

$$\Big\downarrow \ \underline{\text{Newtonian limit}}$$

$$\boxed{\partial_i S^{i}{}_{00} = kT^{i}{}_{0i0} = kT_{00}} \quad (47)$$

The Einstein Hilbert field equation is this limit where:

$$D_\kappa R^{\mu}{}_{\sigma\nu\rho} + D_\kappa R^{\mu}{}_{\rho\sigma\nu} + D_\kappa R^{\mu}{}_{\nu\rho\sigma}$$
$$= k\left(D_\kappa T^{\mu}{}_{\sigma\nu\rho} + D_\kappa T^{\mu}{}_{\rho\sigma\nu} + D_\kappa T^{\mu}{}_{\nu\rho\sigma}\right)$$
$$= 0 \quad (48)$$

and:

9)

$$R^{\mu}{}_{\sigma\nu\rho} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\nu\rho\sigma} = D_{\nu} S^{\mu}{}_{\rho\sigma} + D_{\sigma} S^{\mu}{}_{\nu\rho} + D_{\rho} S^{\mu}{}_{\sigma\nu}$$
$$= 0 \quad -(49)$$

In form notation:

$$R^{a}{}_{b} \wedge q^{b} = (DS)^{a}{}_{b} \wedge q^{b} = 0 \quad -(50)$$

where:

$$(DS)^{a}{}_{b} = D_{\nu} S^{a}{}_{\rho b} \quad -(51)$$

Therefore $S^{a}{}_{\rho b}$ is a differential form analogous to the spin connection $\omega^{a}{}_{\rho b}$.

More generally:

$$(DS)^{a}{}_{b} \wedge q^{b} := D \wedge T^{a} \quad -(52)$$

4) The main advantage of using $S^{a}{}_{b}$ is that there is no need to expand the Christoffel symbol in terms of the metric. This can be seen from eqs. (37) and (38).

5) The Bianchi identity is:

$$\boxed{(DS)^{a}{}_{b} \wedge q^{b} := D \wedge T^{a}} \quad -(53)$$