93(16): Coulomb Law, 4 / 9 / 07

! This calculation re-checks some previous notes, a successful recheck is denoted ✓✓. Also it refers to Carroll's chapter 7 equation numbers. We checked by hand and computer that Carroll gives the right results.

✓✓

We have:

$$R^0{}_{101} = -2(\partial_1\alpha)^2 - \partial_1(\partial_1\alpha) \quad -(1)\ ((7.15))$$

$$R^0{}_{202} = -re^{-2\beta}\partial_1\alpha \quad ✓✓ \quad -(2)\ ((7.15))$$

$$R^0{}_{303} = R^0{}_{202}\sin^2\theta \quad ✓✓ \quad -(3)\ ((7.15))$$

Here:

$$\partial_0\alpha = \partial_0\beta = 0 \quad -(4)\ ((7.17))$$

$$\alpha = -\beta \quad -(5)\ ((7.22))$$

$$e^{2\alpha} = 1 - x \quad -(6)\ ✓✓\ ((7.25\ \&\ (7.29))$$

$$2r\partial_1\alpha + 1 = e^{-2\alpha} \quad -(7)\ ((7.23))$$

So:

$$\partial_1\alpha = \frac{1}{2r}\left(e^{-2\alpha} - 1\right) = \frac{1}{2r}\left(\frac{1}{1-x} - 1\right) \quad -(8)$$

$$= \frac{1}{2r}\left(\frac{x}{1-x}\right) \quad ✓✓$$

So:

$$\boxed{\begin{aligned} e^{2\alpha} &= e^{-2\beta} = 1 - x \\ \partial_1\alpha &= -\partial_1\beta = \frac{1}{2r}\left(\frac{x}{1-x}\right) \end{aligned}} \quad ✓✓ \quad -(9)$$

So:

$$R^0{}_{202} = -re^{-2\beta}\partial_1\alpha$$

$$= -re^{2\alpha}\partial_1\alpha = -r(1-x)\frac{1}{2r}\frac{x}{1-x}$$

$$\boxed{R^0{}_{202} = -\frac{x}{2}} \quad ✓✓ \quad -(10)$$

2) Therefore:

$$\boxed{R^0{}_{303} = -\frac{x}{2}\sin^2\theta} \quad \checkmark\checkmark \qquad -(11)$$

The third element is:

$$R^0{}_{101} = -2(\partial_1\alpha)^2 - \partial_1(\partial_1\alpha) \qquad -(12)$$

where:

$$\partial_1\alpha = \frac{1}{2r}\left(\frac{x}{1-x}\right). \qquad -(13)$$

So:

$$R^0{}_{101} = -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)^2 + \frac{1}{2r^2}\left(\frac{x}{1-x}\right)$$

$$= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x} - 1\right)$$

Therefore:

$$\boxed{\begin{aligned} R^0{}_{101} &= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x} - 1\right) \\ R^0{}_{202} &= -\frac{x}{2} \\ R^0{}_{303} &= -\frac{x}{2}\sin^2\theta \end{aligned}} \qquad -(12)$$

Finally:

$$R^0{}_1{}^{10} = R^0{}_{101} \qquad -(13)$$

$$R^0{}_2{}^{20} = -R^0{}_2{}^{02} = -g^{00}g^{22}R^0{}_{202} \qquad -(14)$$

$$R^0{}_3{}^{30} = -R^0{}_3{}^{03} = -g^{00}g^{33}R^0{}_{303} \qquad -(15)$$

where

$$g^{00} = -\frac{1}{\ }, \quad g^{11} = 1-x, \qquad -(16)$$

$$g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2 \sin^2\theta} \qquad (17)$$

So:

$$\boxed{\begin{aligned} R^{0}{}_{1}{}^{10} &= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x} - 1\right) \\ R^{0}{}_{2}{}^{20} &= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right) \\ R^{0}{}_{3}{}^{30} &= -\frac{1}{2r^2}\left(\frac{x}{1-x}\right) \end{aligned}} \qquad (18)$$

The Coulomb law is:

$$\underline{\nabla} \cdot \underline{E} = -\phi\left(R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30}\right)$$

$$\boxed{\underline{\nabla} \cdot \underline{E} = \frac{\phi}{2r^2}\left(\frac{x}{1-x}\right)\left(\frac{x}{1-x} + 3\right)} \qquad (19)$$

where

$$x = \frac{2mG}{rc^2} \qquad (20)$$

As $x \to 0$;

$$\boxed{\underline{\nabla} \cdot \underline{E} \to \frac{3\phi}{2}\frac{x}{r^2}} \qquad (21)$$