1) 93(3): Calculation of Christoffel Symbols and the Relativistic Correction to g.

It is important to check the calculation of the Christoffel symbols given by Carroll in chapter 7, because the method of paper 91 and notes 93(1) and 93(2) depends on the Christoffel symbol:

$$S^r_{00} = \Gamma^r_{00} = \Gamma^1_{00} \quad - (1)$$

in Carroll's notation. Unfortunately, the Christoffel symbols given by Carroll in chapter 7 contain obvious errors, e.g. dimensional errors. So they are checked here.

The line element used by Carroll is:

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2 \quad - (2)$$

where:

$$\alpha = -\beta = \alpha(r, t). \quad - (3)$$

The Christoffel symbol is:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \quad - (4)$$

The relevant metrics are:

$$g_{00} = -e^{2\alpha}, \quad g_{11} = e^{2\beta}. \quad - (5)$$

The rule for metrics and inverse metrics is:

$$g_{\mu\nu} g^{\mu\nu} = 4. \quad - (6)$$

The metric is defined as $g_{\mu\nu}$ and the inverse

2) metric as $g^{\mu\nu}$. The metrics in eq (2) are diagonal so:

$$g_{00}g^{00} + g_{11}g^{11} + g_{22}g^{22} + g_{33}g^{33} = 4. \quad -(7)$$

All off diagonal elements are zero. So:

$$g_{00}g^{00} = 1, \quad g_{11}g^{11} = 1. \quad -(8)$$

Thus: $$g^{00} = -e^{-2\alpha}, \quad g^{11} = e^{-2\beta}. \quad -(9)$$

Some Christoffel symbols are now calculated to check Carroll.

1) $$\Gamma^0_{00} = \frac{1}{2}g^{00}\left(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}\right) -(10)$$

$$= \frac{1}{2}g^{00}\partial_0 g_{00}. \quad -(11)$$

Here: $$\partial_0 g_{00} = -\partial_0\left(e^{2\alpha(r,t)}\right) \quad -(11)$$

$$= -2(\partial_0\alpha)e^{2\alpha}$$

Thus: $$\boxed{\Gamma^0_{00} = -\partial_0\alpha} \quad -(12)$$

This is the same as Carroll except for a negative sign.

2) $$\Gamma^0_{01} = \frac{1}{2}g^{0\rho}\left(\partial_0 g_{1\rho} + \partial_1 g_{\rho 0} - \partial_\rho g_{01}\right) -(13)$$

3)

$$= \frac{1}{2} g^{00} \partial_1 g_{00} \quad -(14)$$

Here: $\partial_1 g_{00} = -\partial_1 \left(e^{2\alpha}\right) = -2\left(\partial_1 \alpha\right) e^{2\alpha} \quad -(15)$

So: $\boxed{\Gamma^0_{01} = -\partial_1 \alpha} \quad -(16)$

This is again the same as Carroll except for a negative sign.

3) So there are typographical errors in Carroll's chapter 7. The relevant Christoffel symbol for paper 93 is:

$$\Gamma^1_{00} = \frac{1}{2} g^{1\rho} \left(\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00} \right) -(17)$$

$$\boxed{\Gamma^1_{00} = -\frac{1}{2} g^{11} \partial_1 g_{00}} \quad -(18)$$

Thus:

$$\Gamma^1_{00} = g^{11} \left(\partial_1 \alpha\right) e^{2\alpha} \quad -(19)$$

$$= e^{(2\alpha - 2\beta)} \partial_1 \alpha \quad -(20)$$

This is the same as Carroll's result.

$$\boxed{\Gamma^1_{00} = \exp\left(2\alpha - 2\beta\right) \partial_1 \alpha} \quad -(21)$$

It is found experimentally (NASA Cassini)

4) Rat:

$$g_{00} = -\left(1 - \frac{2mG}{rc^2}\right), \quad g_{11} = \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \quad -(2$$

Thus:

$$g^{11} = 1 - \frac{2Gm}{rc^2} \quad -(23)$$

We have:

$$\partial_1 g_{00} = \frac{\partial g_{00}}{\partial r} = -\frac{2mG}{r^2c^2} \quad -(24)$$

So from eq. (18):

$$\Gamma^1_{00} = \frac{mG}{r^2c^2}\left(1 - \frac{2Gm}{rc^2}\right) \quad -(25)$$

From page 91: $-(26)$

$$\boxed{g := -c^2\Gamma^1_{00} = -\frac{mG}{r^2}\left(1 - \frac{2Gm}{rc^2}\right)}$$

The Newtonian g is:

$$g(\text{Newton}) = -\frac{mG}{r^2}, \quad -(27)$$

$$F = mg \quad -(28)$$

5)

It is seen from eq. (26) that the Newtonian g is corrected by:

$$\boxed{x = 1 - \frac{2GM}{rc^2}} \quad -(29)$$

in the correct S.I. units. Note that Carroll uses reduced units, $c = 1$.

The correction in eq. (29) is the source of ALL the precisely known relativistic effects such as the precession of the perihelion, angle of deflection at closest approach, Shapiro delay, gravitational red-shift, and Lense-Thirring effect.

As can be seen we have derived it entirely from the S tensor, using only one Christoffel symbol.

Assuming that there are no relevant errors in Carroll's chapter 7, the standard geodesic approach gives his equation (7.48):

$$\bar{V}(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{rc^2} + \frac{L^2}{2r^2}\left(1 - \frac{2GM}{rc^2}\right) \quad -(30)$$

where the correct S.I. units have been used.

6) The Newtonian result is:

$$\bar{V}(\text{Newton}) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{rc^2} + \frac{L^2}{2r^2} \quad -(31)$$

So it is seen that:

$$\bar{V}(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{rc^2} + \frac{L^2}{2r^2}\,x \quad -(32)$$

$$\boxed{\bar{V}(r) = \frac{1}{2}\left(\epsilon + \frac{L^2}{r^2}\right)x} \quad -(33)$$

Eq. (32) comes from the constancy of:

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \quad -(34)$$

so ϵ is a constant of motion along a geodesic.

Here:

$$L = r^2\frac{d\phi}{d\lambda}. \quad -(35)$$

For a particle with mass it is the angular momentum per unit mass. For a massless particle:

$$\epsilon = 0 \quad -(36)$$

and L is the angular momentum. So for a massless particle:

$$\boxed{\bar{V}(r) = \frac{x r^2}{2}\left(\frac{d\phi}{d\lambda}\right)^2} \quad -(37)$$

7) Eq. (35) is the general relativistic equivalent of Kepler's second law, (equal areas are swept out in equal times).

So light deflection due to gravity is due to:

$$\boxed{V(r) = \frac{r^2}{2}\left(1 - \frac{2GM}{rc^2}\right)\left(\frac{d\phi}{d\lambda}\right)^2} \quad (38)$$

The Newtonian result for $\epsilon = 0$ from eq (31) is obtained when:

$$x = 1 \quad (39)$$

$$\boxed{V(\text{Newton}, \epsilon = 0) = \frac{r^2}{2}\left(\frac{d\phi}{d\lambda}\right)^2} \quad (40)$$

From eq. (40), the Newtonian deflection is:

$$\delta = \frac{2GM}{r_0 c^2} \quad (41)$$

where r_0 is the distance of closest approach. It is seen by inspection of eq. (38) that in general relativity there is an additional $2GM/rc^2$, so:

$$\boxed{\delta(gr) = \frac{4GM}{r_0 c^2}} \quad (42)$$

observed experimentally by NASA to ...

8) Units Check

The reduced units used by Carroll can be misleading, because c is missing. So to check his units for self-consistency:

$$\epsilon = -g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} \quad - (A1)$$

$$L = r^2\frac{d\phi}{d\lambda}, \quad \lambda = a\tau + b, \quad - (A2)$$

so: $\epsilon = m^2 s^{-2}, \quad L = m^2 s^{-1}, \quad - (A3)$

so: $\frac{L^2}{r^2} = m^4 s^{-2} m^{-2} = m^2 s^{-2}$ ✓

so Carroll's eqn. (30) is dimensionally consistent, but $V(r)$ is given the units of $(m/s)^2$. This is not dimensionally correct, $V(r)$ must have the units of energy: because

$$\text{Energy} = \text{joules} = \text{kgm}(m/s)^2 \quad - (A4)$$

so $V(r)$ has the units of energy per unit mass.

I think that Carroll does point this out, but these notes give the correct units.