98(1): Relation between the Conserved Noether Current of Angular Momentum and the Cartan Torsion

In Minkowski space-time it is well known that the Noether Theorem: there is a conserved current from the Noether Theorem:

$$J^{\mu}{}_{\rho\sigma} = -T^{\mu}{}_{\kappa} X^{\kappa}{}_{\rho\sigma} \quad -(1)$$

where $T^{\mu}{}_{\kappa}$ is the canonical energy momentum ~~and~~ density tensor and:

$$X^{\mu}{}_{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu}_{\rho} x_{\sigma} - \delta^{\mu}_{\sigma} x_{\rho}\right) \quad -(2)$$

This is the result of invariance under spatial rotations in Minkowski space-time. The Noether Theorem is basic to physics and the existence of $J^{\mu}{}_{\rho\sigma}$ is a fundamental justification for defining the ECE electromagnetic field:

$$F^{\mu}{}_{\rho\sigma} = A^{(0)} T^{\mu}{}_{\rho\sigma} \quad -(3)$$

in terms of the Cartan torsion in a space-time that is curving and spinning. It is seen that both $J^{\mu}{}_{\rho\sigma}$ and $F^{\mu}{}_{\rho\sigma}$ are rank three tensors, and one must be proportional to the other because the electromagnetic field is

2) known experimentally to have angular momentum.

In flat space-time:

$$J^{\mu}{}_{\rho\sigma} = -\frac{1}{2}\left(T^{\mu}{}_{\rho}x_{\sigma} - T^{\mu}{}_{\sigma}x_{\rho}\right) \quad -(4)$$

(L. H. Ryder, "Quantum Field Theory" (CUP, 2nd ed., 1996). Eq. (4) has page 89,

components such as:

orbital

$$J^{1}{}_{01} = -\frac{1}{2}\left(T^{1}{}_{0}x_{1} - T^{1}{}_{1}x_{0}\right) \quad -(5)$$

$$J^{2}{}_{02} = -\frac{1}{2}\left(T^{2}{}_{0}x_{2} - T^{2}{}_{2}x_{0}\right) \quad -(6)$$

$$J^{3}{}_{03} = -\frac{1}{2}\left(T^{3}{}_{0}x_{3} - T^{3}{}_{3}x_{0}\right) \quad -(7)$$

spin

$$J^{1}{}_{23} = -\frac{1}{2}\left(T^{1}{}_{2}x_{3} - T^{1}{}_{3}x_{2}\right) \quad -(8)$$

$$J^{2}{}_{31} = -\frac{1}{2}\left(T^{2}{}_{3}x_{1} - T^{2}{}_{1}x_{3}\right) \quad -(9)$$

$$J^{3}{}_{12} = -\frac{1}{2}\left(T^{3}{}_{1}x_{2} - T^{3}{}_{2}x_{1}\right) \quad -(10)$$

These have the same structure as the electric and magnetic fields of ECE theory, but without the spin connection:

$$E^{1}{}_{01} = \partial_0 A^{1}{}_{1} - \partial_1 A^{1}{}_{0} \quad -(11)$$

$$E^{2}{}_{02} = \partial_0 A^{2}{}_{2} - \partial_2 A^{2}{}_{0} \quad -(12)$$

$$E^{3}{}_{03} = \partial_0 A^{3}{}_{3} - \partial_3 A^{3}{}_{0} \quad -(13)$$

3)

$$B^1{}_{23} = \partial_2 A^1{}_3 - \partial_3 A^1{}_2 \quad -(14)$$

$$B^2{}_{31} = \partial_3 A^2{}_1 - \partial_1 A^2{}_3 \quad -(15)$$

$$B^3{}_{12} = \partial_1 A^3{}_2 - \partial_2 A^3{}_1 \quad -(16)$$

The spin connection is missing in eqns (5) to (10) because they are derived in flat spacetime. In ECE theory it is known that the spin connection must be added, so eq. (14) for example becomes:

$$B^1{}_{23} = \partial_2 A^1{}_3 - \partial_3 A^1{}_2 + \omega^1{}_{2b} A^b{}_3 - \omega^1{}_{3b} A^b{}_2 \quad -(17)$$

$$\boxed{\text{However it is clear that the Cartan torsion } T^\mu{}_{\rho\sigma} \text{ is a generalization of } J^\mu{}_{\rho\sigma} \text{ to a space-time with curvature and torsion.}}$$

The units of $T^\mu{}_{\rho\sigma}$ are m^{-1} and those of $J^\mu{}_{\rho\sigma}$ are of angular momentum density i.e. $J s m^{-3}$. Therefore the units of the ratio of $J^\mu{}_{\rho\sigma}$ to $T^\mu{}_{\rho\sigma}$ are $J s m^{-2}$. These are the units of $eB^{(0)}$ or $eE^{(0)}/c$

where $B^{(0)}$ is magnetic flux density in $J\,s\,m^{-2}\,C^{-1}$ and e is electric charge in C. So:

$$\boxed{J^{\mu}{}_{\rho\sigma} = \frac{eE^{(0)}}{c}\,T^{\mu}{}_{\rho\sigma}} \quad -(18)$$

Here $-e$ may be taken as the charge on the electron.

<u>Electric Field Strength</u> $\quad -(19)$

$$E^{i}{}_{0i} = E^{(0)}\,T^{i}{}_{0i}, \quad i = 1, 2, 3.$$

<u>Magnetic Flux Density</u>

$$B^{i}{}_{jk} = B^{(0)}\,T^{i}{}_{jk}, \quad i, j, k = 1, 2, 3 \quad -(20)$$

It is seen that the electric field strength and magnetic flux density of ECE are components of the fundamental Noether current $J^{\mu}{}_{\rho\sigma}$ in a space-time with both torsion and curvature. This is a conserved Noether current:

$$D_{\mu}\,J^{\mu}{}_{\rho\sigma} = 0 \quad -(20)$$

in general relativity.

5) It follows from eq (18) that for a constant $E^{(0)}$:

$$D_\mu T^\mu{}_{\rho\sigma} = 0 \quad -(21)$$

where

$$T^\mu{}_{\rho\sigma} = q^\mu{}_a T^a{}_{\rho\sigma} \quad -(22)$$

where $T^a{}_{\rho\sigma}$ is the torsion form of differential geometry. Eq. (21) means, in flat spacetime

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(23)$$

and

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(24)$$

This is true in a flat spacetime, where

$$D_\mu T^\mu{}_{\rho\sigma} \rightarrow \partial_\mu T^\mu{}_{\rho\sigma} = 0, \quad -(25)$$

otherwise the spin connection must be considered. However this is enough to show that eq (21) refers to the field in the absence of a source. This means that angular momentum density is conserved in the absence of a torque. Total angular momentum density is always conserved.