

Derivation of the Quantum Hamilton Equations of Motion and refutation of the Copenhagen interpretation.

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Abstract.

The quantized versions of the Hamilton equations of motion are derived straightforwardly. The derivation is a refutation of the Copenhagen interpretation of quantum mechanics because in the quantum Hamilton equations, position and momentum are specified simultaneously. The Schroedinger equation of motion is derived from the ECE wave equation through use of concepts associated with the ECE fermion equation, and a novel anti-commutator method is developed to refute the Copenhagen interpretation in other ways. This leaves the ECE interpretation of quantum mechanics as part of the only valid unified field theory known at present.

Keywords: ECE theory, quantum Hamilton equations, refutation of Copenhagen, anti commutator method.

1. Introduction.

As part of this series of papers [1–10] developing the Einstein Cartan Evans (ECE) unified field theory, the fermion equation of motion of relativistic quantum mechanics has been derived in the three preceding papers (UFT 172 to UFT 174 on www.aias.us). In Section 2, the methods used to develop the fermion equation are used to show that in the non-relativistic limit, the Schroedinger equation of motion derives from differential geometry within the philosophy of relativity. The Schroedinger axiom that defines the operators of quantum mechanics therefore originates in geometry together with the concept of quantum classical equivalence, that equations of quantum mechanics must correspond in a well defined limit to equations of classical mechanics. There are well known exceptions [11] in which purely quantum phenomena exist, but quantum classical equivalence is the rule rather than the exception. Therefore it is shown that the fundamental axioms of quantum mechanics can be derived from geometry and relativity.

In Section 3, the newly derived operators are used to infer the existence of the quantized equivalents of the Hamilton equations of motion, which Hamilton derived in about 1833 without the use of the lagrangian dynamics. It is well known that the Hamilton equations use position (x) and momentum (p) as conjugate variables in the well defined classical sense [12], and so x and p are “specified simultaneously” in the dense Copenhagen jargon of the twentieth century. Therefore, by quantum classical equivalence, x and p are specified simultaneously in the quantum Hamilton equations, thus refuting the Copenhagen interpretation of quantum mechanics based on the commutator $[\hat{x}, \hat{p}]$ of the operators \hat{x} and \hat{p} . The quantum Hamilton equations show that x and p are specified simultaneously in quantum mechanics, and are hitherto unknown equations first derived in this paper. It took a long time (1926 – 2011) to derive the quantum Hamilton equations because it was thought incorrectly that x and p cannot be specified simultaneously in quantum mechanics. This is a clear illustration of the negative effect that Copenhagen has had on the development of the subject.

In Section 4, the anticommutator $\{\hat{x}, \hat{p}\}$ is used to derive further refutations of Copenhagen, in that $\{\hat{x}, \hat{p}\}$ acting on wavefunctions that are exact solutions of Schroedinger’s equation produces expectation values that are zero for the harmonic oscillator, and non zero for atomic H. The anticommutator $\{\hat{x}, \hat{p}\}$ is shown to be proportional to the commutator $[\hat{x}^2, \hat{p}^2]$, whose expectation values for the harmonic oscillator are all zero, while for atomic H they are all non-zero. For the particle on a ring [11] combinations can be zero, while individual commutators of this type are non-zero. For linear motion self inconsistencies in Copenhagen are revealed, and for the particle on a sphere the commutator is again non-zero. The hand calculations (see fifteen detailed calculation notes accompanying this paper, UFT 175, on www.aias.us) are checked with computer algebra which is used to produce tables of relevant expectation values. Copenhagen is refuted because in that interpretation it makes no sense for an expectation value of a commutator of operators to be both zero and non-zero for the same pair of operators. One of the operators could be “absolutely unknowable” and the other “precisely knowable” if the expectation value were non-zero, and both “precisely knowable” if it were zero. These two interpretations refer respectively to non-zero and zero commutator expectation values, and both interpretations cannot be true for the same pair of operators. Prior to this paper, commutators of a given pair of operators were thought to be either zero or non-zero, never both zero and non-zero, so a clear refutation of Copenhagen was never realized. In ECE quantum mechanics, Copenhagen and its unscientific jargon are not used, and the expectation values are straightforward consequences of the fundamental operators introduced by Schroedinger. The latter immediately rejected Copenhagen as did Einstein, de Broglie and their Schools.

2. Derivation of the Schroedinger equation from geometry.

The clearest route to the derivation of the Schroedinger equation from Cartan geometry [1–10] was given in the preceding paper (UFT 174) and is briefly reviewed here for ease of reference. The starting point is the most fundamental property of Cartan geometry: the complete vector field in any dimension is independent of how it is built up with basis elements and components. In three

dimensions for example a vector field is the same vector field in Cartesian and spherical polar coordinates or any curvilinear coordinate system. In differential geometry this property is known as the tetrad postulate [1–10].

$$D_\mu q_\nu^a = \partial_\mu q_\nu^a + \omega_{\mu b}^a q_\nu^b - \Gamma_{\mu\nu}^\lambda q_\lambda^a = 0 \quad (1)$$

where q_ν^a is Cartan's tetrad and where D_μ is the covariant derivative. The three index object $\omega_{\mu b}^a$ is the spin connection. Using:

$$\omega_{\mu\nu}^a = \omega_{\mu b}^a q_\nu^b \quad , \quad \Gamma_{\mu\nu}^a = \Gamma_{\mu\nu}^\lambda q_\lambda^a \quad , \quad (2)$$

the tetrad postulate may be expressed as:

$$\partial_\mu q_\nu^a = \Omega_{\mu\nu}^a = \Gamma_{\mu\nu}^a - \omega_{\mu\nu}^a \quad . \quad (3)$$

Application of ∂^μ gives the ECE wave equation as follows:

$$(\square + R) q_\nu^a = 0 \quad . \quad (4)$$

The scalar valued object R is defined by:

$$R := q_a^\nu (\omega_{\mu\nu}^a - \Gamma_{\mu\nu}^a) \quad . \quad (5)$$

In these derivations, $\Gamma_{\mu\nu}^\lambda$ is the Riemann connection. The ECE wave equation is a fundamental wave equation of geometry from which may be derived all the well known wave equations of physics, and perhaps some that are not yet known.

The fermion equation in wave format is a limit of the wave equation (4) when:

$$R \longrightarrow \left(\frac{mc}{\hbar}\right)^2 \quad (6)$$

and in SU(2) representation space. In this space the fermion spinor is:

$$\psi = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} . \quad (7)$$

as in UFT 172 to UFT 174 on www.aias.us. The fermion equation reduces as in UFT 174 to:

$$\hat{H} \psi = E \psi \quad , \quad (8)$$

which is a relativistic format of Schroedinger's equation. The latter is obtained using the minimal prescription for the total energy E :

$$E \longrightarrow E - e\varphi \quad (9)$$

where $-e$ is the charge on the electron and where φ is the Coulomb potential in the case of atomic and molecular spectra. In the absence of a vector potential the hamiltonian operator \hat{H} is:

$$\hat{H} = m c^2 + e\varphi + \frac{\hat{p}^2}{2m} \quad (10)$$

so:

$$\frac{\hat{p}^2}{2m} \psi = (E - m c^2 - e\varphi) \psi \quad . \quad (11)$$

Schroedinger's axiom is:

$$\hat{p} \psi = -i \hbar \nabla \psi \quad (12)$$

so Eq. (11) becomes:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = (E - m c^2 - e\varphi) \psi \quad (13)$$

In the non-relativistic limit:

$$E - m c^2 \longrightarrow T = \frac{1}{2} m v^2 \quad (14)$$

and in this limit the Schroedinger equation is obtained:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = (T + V) \psi \quad (15)$$

Note that in this derivation the fundamental axiom (12) of quantum mechanics follows from the wave equation (4) and from the necessity that the classical equivalent of the hamiltonian operator \hat{H} is the hamiltonian in classical dynamics, the sum of the kinetic and potential energies:

$$H = T + V \quad (16)$$

So in ECE physics, quantum mechanics can be derived from general relativity in a straightforward way that can be tested against experimental data at each stage.

3. Derivation of the quantum Hamilton equations.

The two quantum Hamilton equations are derived respectively using the well known position and momentum representations of quantum mechanics [11]. In the position representation the Schroedinger axiom is:

$$\hat{p} \psi = -i \hbar \frac{\partial}{\partial x} \psi, \quad (\hat{p} \psi)^* = i \hbar \frac{\partial}{\partial x} \psi^*, \quad (17)$$

from which it follows that:

$$[\hat{x}, \hat{p}] \psi = i \hbar \psi \quad (18)$$

so the expectation value of the commutator is:

$$\langle [\hat{x}, \hat{p}] \rangle = i \hbar \quad (19)$$

In the position representation the expectation value $\langle x \rangle$ is x . It follows that:

$$\frac{d}{dx} \langle \hat{x} \rangle = \frac{i}{\hbar} \langle [\hat{x}, \hat{p}] \rangle = 1 \quad (20)$$

Now note that this tautology can be derived as follows:

$$\frac{d}{dx} \langle \hat{x} \rangle = -\frac{d}{dx} \int \psi^* \hat{x} \psi d\tau \quad (21)$$

The proof of Eq. (21) is straightforward and is as follows. First use the Leibniz Theorem to find:

$$-\frac{d}{dx} \int \psi^* \hat{x} \psi d\tau = -\left(\int \frac{d\psi^*}{dx} \hat{x} \psi d\tau + \int \psi^* \hat{x} \frac{d\psi}{dx} d\tau \right) \quad (22)$$

In quantum mechanics the operators are hermitian operators that are defined as follows:

$$\int \psi_m^* \hat{A} \psi_n d\tau = (\int \psi_n^* \hat{A} \psi_m d\tau)^* = (\int \hat{A}^* \psi_m^* \psi_n d\tau) . \quad (23)$$

Therefore it follows that Eq. (22) is:

$$\frac{d}{dx} \langle \hat{x} \rangle = 1 = \frac{i}{\hbar} \int \psi^* (\hat{p} \hat{x} - \hat{x} \hat{p}) \psi d\tau = \frac{i}{\hbar} \langle [\hat{p}, \hat{x}] \rangle \quad (24)$$

which is Eq. (20), Q.E.D.

The first quantum Hamilton equation is obtained by generalizing \hat{x} to any hermitian operator \hat{A} of quantum mechanics:

$$\hat{x} \longrightarrow \hat{A} \quad (25)$$

so one format of the first quantum Hamilton equation is:

$$\frac{d}{dx} \langle \hat{A} \rangle = \frac{i}{\hbar} \langle [\hat{p}, \hat{A}] \rangle \quad (26)$$

In the special case:

$$\hat{A} = \hat{H} , \quad (27)$$

then:

$$\frac{d}{dx} \langle \hat{H} \rangle = \frac{i}{\hbar} \langle [\hat{p}, \hat{H}] \rangle \quad (28)$$

However, it is known that [11]:

$$\frac{d}{dt} \langle \hat{p} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{p}] \rangle \quad (29)$$

so from Eqs. (28) and (29) the quantum Hamilton equation is:

$$\frac{d}{dx} \langle \hat{H} \rangle = - \frac{d}{dt} \langle \hat{p} \rangle . \quad (30)$$

The expectation values in this equation are:

$$H = \langle \hat{H} \rangle , \quad p = \langle \hat{p} \rangle . \quad (31)$$

so the first Hamilton equation of motion of 1833 follows, Q.E.D.:

$$\frac{dH}{dx} = - \frac{dp}{dt} . \quad (32)$$

The second quantum Hamilton equation follows from the momentum representation {11}:

$$\hat{x} \psi = - \frac{\hbar}{i} \frac{\partial \psi}{\partial p} , \quad \hat{p} \psi = p \psi , \quad (33)$$

from which the following tautology follows:

$$\frac{d}{dp} \langle \hat{p} \rangle = - \frac{i}{\hbar} \langle [\hat{x}, \hat{p}] \rangle = 1 . \quad (34)$$

This tautology can be obtained from the equation:

$$\frac{d}{dp} \langle \hat{p} \rangle = -\frac{d}{dp} \int \psi^* \hat{p} \psi d\tau . \quad (35)$$

Now generalize \hat{p} to any operator \hat{A} :

$$\hat{p} \longrightarrow \hat{A} \quad (36)$$

and the second quantum Hamilton equation in one format is:

$$\frac{d}{dp} \langle \hat{A} \rangle = -\frac{i}{\hbar} \langle [\hat{x}, \hat{A}] \rangle . \quad (37)$$

In the special case:

$$\hat{A} = \hat{H} , \quad (38)$$

the second quantum Hamilton equation is:

$$\frac{d}{dp} \langle \hat{H} \rangle = -\frac{i}{\hbar} \langle [\hat{x}, \hat{H}] \rangle . \quad (39)$$

However, it is known that:

$$\langle [\hat{x}, \hat{H}] \rangle = -\frac{\hbar}{i} \frac{d}{dt} \langle \hat{x} \rangle , \quad (40)$$

so the second quantum Hamilton equation is:

$$\frac{d}{dp} \langle \hat{H} \rangle = \frac{d}{dt} \langle \hat{x} \rangle \quad (41)$$

which is the second Hamilton equation of classical dynamics, Q.E.D.:

$$\frac{dH}{dp} = \frac{dx}{dt} . \quad (42)$$

Note carefully that both the quantum Hamilton equations derive directly from the familiar commutator (18) of quantum mechanics. Conversely the Hamilton equations of 1833 imply the commutator (18) given only the Schroedinger postulate in position and momentum representation respectively. In the Hamilton equations of classical dynamics, x and p are simultaneously observable, so they are also simultaneously observable in the quantized Hamilton equations of motion and in quantum mechanics in general. This argument refutes Copenhagen, in which x and p are asserted not to be simultaneously observable. The Copenhagen interpretation is an unscientific fallacy and should be discarded. The important result of this section is the first derivation of the quantum Hamilton equations of motion. These can be implemented in a large number of ways.

4. The anticommutator method of refuting Copenhagen.

Consider the anticommutator:

$$\{\hat{x}, \hat{p}\} \psi = (\hat{p} \hat{x} + \hat{x} \hat{p}) \psi . \quad (43)$$

In the position representation it may be developed as follows:

$$\{\hat{x}, \hat{p}\} \psi = -i \hbar \left(x \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial x} (x \psi) \right) = -i \hbar \left(\psi + 2x \frac{\partial \psi}{\partial x} \right) . \quad (44)$$

The commutator of \hat{p}^2 and \hat{x}^2 is defined [11] as:

$$[\hat{x}^2, \hat{p}^2] \psi = ([\hat{x}^2, \hat{p}] \hat{p} + \hat{p} ([\hat{x}^2, \hat{p}]) \psi . \quad (45)$$

Now use the quantum Hamilton equations to find that:

$$[\hat{p}, \hat{x}^2] \psi = -2 i \hbar x \psi \quad (46)$$

$$[\hat{x}^2, \hat{p}] \psi = 2 i \hbar x \psi . \quad (47)$$

It follows that:

$$[\hat{x}^2, \hat{p}^2] \psi = 2 i \hbar (\hat{p} \hat{x} + \hat{x} \hat{p}) \psi . \quad (48)$$

So the following useful equation has been proven in one dimension:

$$[\hat{x}^2, \hat{p}^2] \psi = 2 i \hbar \{ \hat{x}, \hat{p} \} \psi . \quad (49)$$

In three dimensions the Schroedinger axiom in position representation is:

$$\hat{p} \psi = -i \hbar \nabla \psi , \quad (50)$$

and in three dimensions the relevant commutator is:

$$[\hat{r}, \hat{p}] \psi = -i \hbar (\mathbf{r} \cdot \nabla \psi - \nabla \cdot (\mathbf{r} \psi)) \quad (51)$$

where in Cartesian coordinates:

$$\mathbf{r}^2 = X^2 + Y^2 + Z^2 . \quad (52)$$

Therefore:

$$[\hat{r}, \hat{p}] \psi = -i \hbar (\mathbf{r} \cdot \nabla \psi - \psi \nabla \cdot \mathbf{r} - \mathbf{r} \cdot \nabla \psi) \quad (53)$$

where:

$$\nabla \cdot (\mathbf{r} \psi) = \psi \nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla \psi \quad (54)$$

in which:

$$\nabla \cdot \mathbf{r} = 3 . \quad (55)$$

So:

$$[\hat{r}, \hat{p}] \psi = 3 i \hbar \psi . \quad (56)$$

In three dimensions:

$$[\hat{r}^2, \hat{p}^2] \psi = ([\hat{r}^2, \hat{p}] \cdot \hat{p} + \hat{p} ([\hat{r}^2, \hat{p}]) \psi . \quad (57)$$

where:

$$\begin{aligned} [\hat{r}^2, \hat{p}] \psi &= r^2 \hat{p} \psi - \hat{p} (r^2 \psi) \\ &= i \hbar \nabla r^2 \psi \end{aligned} \quad (58)$$

and where:

$$\nabla r^2 = \frac{\partial r^2}{\partial x} \mathbf{i} + \frac{\partial r^2}{\partial y} \mathbf{j} + \frac{\partial r^2}{\partial z} \mathbf{k} \quad (59)$$

with:

$$r^2 = X^2 + Y^2 + Z^2 \quad (60)$$

So:

$$\nabla r^2 = 2 \mathbf{r} \quad (61)$$

and the three dimensional equivalent of Eq. (49) is:

$$[\hat{r}^2, \hat{p}^2] \psi = 2 i \hbar \{ \mathbf{r}, \mathbf{p} \} \psi \quad (62)$$

The anticommutator in this equation is:

$$\begin{aligned} (\hat{r} \cdot \hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \psi &= \mathbf{r} \cdot \hat{\mathbf{p}} \psi + \hat{\mathbf{p}} \cdot (\mathbf{r} \psi) \\ &= -i \hbar (2 \mathbf{r} \cdot \nabla \psi + 3 \psi) \end{aligned} \quad (63)$$

where:

$$\mathbf{r} \cdot \nabla \psi = X \frac{\partial \psi}{\partial X} + Y \frac{\partial \psi}{\partial Y} + Z \frac{\partial \psi}{\partial Z} \quad (64)$$

so in Cartesian coordinates:

$$\{ \hat{r}, \hat{\mathbf{p}} \} \psi = -i \hbar (2 (X \frac{\partial \psi}{\partial X} + Y \frac{\partial \psi}{\partial Y} + Z \frac{\partial \psi}{\partial Z}) + 3 \psi) \quad (65)$$

When considering the H atom the relevant anticommutator is:

$$\{ \hat{r}, \hat{p}_r \} \psi = -i \hbar \left\{ r, \frac{\partial}{\partial r} \right\} \psi \quad (66)$$

so we obtain:

$$\{ \hat{r}^2, \hat{p}_r^2 \} \psi = 2 \hbar^2 \left(\psi + 2 r \frac{\partial \psi}{\partial r} \right) \quad (67)$$

With these basic definitions some expectation values:

$$\langle [\hat{r}^2, \hat{p}_r^2] \rangle = 2 i \hbar \langle \{ \hat{r}, \hat{\mathbf{p}} \} \rangle \quad (68)$$

are worked out for exact solutions of the Schroedinger equation in the fifteen calculation notes accompanying this paper (UFT 175 on www.aias.us). These expectation values have been checked by computer algebra and are given in Tables in the following section of this paper. The result is obtained that the expectation values can be zero or non-zero depending on which solution of Schroedinger's equation is used. This result refutes the Copenhagen interpretation for reasons given already in this paper. If this result had been known in 1926, when Schroedinger inferred his equation, the unscientific fallacy of Copenhagen would not have arisen due to the Solvay Conference of 1927. Unfortunately, a voluminous literature has appeared on the so called Heisenberg uncertainty principle, this literature should be discarded in favour of ECE quantum mechanics.

5. Tables of expectation values.

In this section some tables of expectation values are given for the well known exact solutions [11] of the Schroedinger equation: linear motion, the harmonic oscillator, particle on a ring and on a sphere, and the H atom. In Tables 1-5 the values for the commutators and anti-commutators are given. As can be seen, the expectation value of the commutator of generalized coordinates $\langle [\hat{q}, \hat{p}] \rangle$ is always $i \hbar$ which is the well known Heisenberg uncertainty. For the harmonic oscillator, the

commutator of squared operators $\{\hat{q}^2, \hat{p}^2\}$ is zero, however. The same holds for the radial wave functions of the H atom. This shows the inconsistency of the Heisenberg concept. Obviously, q or p cannot be “unknowable” and also “knowable” as in the claim made originally by Bohr and Heisenberg independently circa 1927.

In Table 4 it can be seen that the angular momentum operator \hat{L}_Z commutes with \hat{L}^2 as is well known. The forms of these operators in spherical coordinates are given in [11] in full detail. For most cases with angle coordinates, the sinus values of the angle give vanishing commutators, but not the angle values themselves. The results show that an individual treatment is required for each solution of the Schroedinger equation.

n	$[x, p_x]$	$\{x, p_x\}$	$[x, p_x^2]$	$\{x, p_x^2\}$	$[x^2, p_x]$	$\{x^2, p_x\}$	$[x^2, p_x^2]$	$\{x^2, p_x^2\}$
0	$i\hbar$	0	0	0	0	0	0	$-\frac{\hbar^2}{2}$
1	$i\hbar$	0	0	0	0	0	0	$\frac{3\hbar^2}{2}$
2	$i\hbar$	0	0	0	0	0	0	$\frac{11\hbar^2}{2}$
3	$i\hbar$	0	0	0	0	0	0	$\frac{23\hbar^2}{2}$

Table 1. Commutators for harmonic oscillator.

$[\psi, l_z]$	$\{\psi, l_z\}$	$[\psi, l_z^2]$	$\{\psi, l_z^2\}$	$[\psi^2, l_z]$	$\{\psi^2, l_z\}$	$[\psi^2, l_z^2]$	$\{\psi^2, l_z^2\}$
$i\hbar$	$\hbar(2\pi m - i)$	$2im\hbar^2$	$2m\hbar^2(\pi m - i)$	$2i\pi\hbar$	$\frac{2}{3}\pi\hbar(4\pi m - 3i)$	$2\hbar^2(2i\pi m + 1)$	$\frac{2}{3}\hbar^2(4\pi^2 m^2 - 6i\pi m - 3)$

Table 2. Commutators for a particle on a ring.

$[\sin(\psi), L_z]$	$\{\sin(\psi), L_z\}$	$[\sin(\psi), L_z^2]$	$\{\sin(\psi), L_z^2\}$	$[\sin^2(\psi), L_z]$	$\{\sin^2(\psi), L_z\}$	$[\sin^2(\psi), L_z^2]$	$\{\sin^2(\psi), L_z^2\}$
0	0	0	0	0	$m\hbar$	0	$m^2\hbar^2$

Table 3. Additional commutators for a particle on a ring.

l, m	$[\psi, L^2]$	$\{\psi, L^2\}$	$[\sin(\psi), L^2]$	$\{\sin(\psi), L^2\}$	$[\theta, L^2]$	$\{\theta, L^2\}$	$[\sin(\theta), L^2]$	$\{\sin(\theta), L^2\}$	$[L_z, L^2]$	$\{L_z, L^2\}$
0, 0	0	0	0	0	0	0	0	0	0	0
1, 0	0	$4\pi\hbar^2$	0	0	0	$2\pi\hbar^2$	0	$\frac{3}{4}\pi\hbar^2$	0	0
1, 1	$3i\hbar^2$	$(4\pi - 3i)\hbar^2$	0	0	0	$2\pi\hbar^2$	0	$\frac{9}{8}\pi\hbar^2$	0	$4\hbar^3$
2, 0	0	$12\pi\hbar^2$	0	0	0	$6\pi\hbar^2$	0	$\frac{75}{32}\pi\hbar^2$	0	0
2, 1	$5i\hbar^2$	$(12\pi - 5i)\hbar^2$	0	0	0	$6\pi\hbar^2$	0	$\frac{45}{16}\pi\hbar^2$	0	$12\hbar^3$
2, 2	$5i\hbar^2$	$(12\pi - 5i)\hbar^2$	0	0	0	$6\pi\hbar^2$	0	$\frac{225}{64}\pi\hbar^2$	0	$24\hbar^3$

Table 4. Commutators for spherical harmonics.

l, m	$[r, p_r]$	$\{r, p_r\}$	$[r, p_r^2]$	$\{r, p_r^2\}$	$[r^2, p_r]$	$\{r^2, p_r\}$	$[r^2, p_r^2]$	$\{r^2, p_r^2\}$
0, 0	$i\hbar$	$2i\hbar$	0	$\frac{1}{a_0}\hbar^2$	$3ia_0\hbar$	$3ia_0\hbar$	0	0
1, 0	$i\hbar$	$2i\hbar$	0	$\frac{1}{a_0}\hbar^2$	$12ia_0\hbar$	$12ia_0\hbar$	0	$3\hbar^2$
1, 1	$i\hbar$	$2i\hbar$	0	$\frac{1}{2a_0}\hbar^2$	$10ia_0\hbar$	$10ia_0\hbar$	0	\hbar^2
2, 0	$i\hbar$	$2i\hbar$	0	$\frac{1}{a_0}\hbar^2$	$27ia_0\hbar$	$27ia_0\hbar$	0	$8\hbar^2$
2, 1	$i\hbar$	$2i\hbar$	0	$\frac{7}{9a_0}\hbar^2$	$25ia_0\hbar$	$25ia_0\hbar$	0	$6\hbar^2$
2, 2	$i\hbar$	$2i\hbar$	0	$\frac{1}{3a_0}\hbar^2$	$21ia_0\hbar$	$21ia_0\hbar$	0	$2\hbar^2$

Table 5. Commutators for radial wave functions of the hydrogen atom. a_0 is the Bohr radius.

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