Development of the quantum Hamilton equations.

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Abstract.

The quantum Hamilton equations inferred in UFT 175 are developed for use with higher order commutators and a useful new equation of motion of quantum mechanics inferred from them. This equation may be used to find a wavefunction from a given classical hamiltonian and is a first order equation that is easier to solve analytically or numerically than the second order Schroedinger equation.

Keywords: ECE theory, quantum Hamilton equations, higher order commutators, new equation of quantum mechanics.

1.Introduction.

The development of a relatively simple unified field theory such as the Einstein Cartan Evans (ECE) unified field theory [1-10] makes available for the first time in physics a deterministic and relativistic quantum mechanics which does not utilize the Copenhagen interpretation. The central equation of ECE quantum mechanics is the fermion equation developed in UFT 172 onwards (www.aias.us). This equation is deduced from the tetrad postulate of Cartan geometry and removes the concept of negative energy from quantum field theory. The latter is therefore in need of extensive revision because a multi particle interpretation of the equation of fermions of all kinds is no longer needed. During the course of the development from UFT 172 onwards the Pauli exclusion principle has been deduced from the fermion equation and the Heisenberg uncertainty principle refuted straightforwardly. In UFT 175 new equations of quantum mechanics were inferred, and named the quantum Hamilton equations.

In Section 2, a new equation of motion of quantum mechanics is deduced directly from the quantum Hamilton equations. Using this equation a wavefunction can be deduced from a given classical hamiltonian and without use of the second order hamiltonian operator of the Schroedinger equation. In the case of time independent wavefunction this equation assumes a simple format, and is generally applicable throughout quantum mechanics. The quantum Hamilton equations of UFT 175 are put into a canonical format that is directly comparable with their well known classical counterparts.

In Section 3, higher order quantum Hamilton equations are developed straightforwardly with fundamental commutator algebra, and the methods of this section are illustrated with the second order commutator of position x and momentum p in the one dimensional Cartesian representation. This commutator is derived self consistently in position representation. The reduction of the quantum Hamilton equations to the corresponding classical Poisson bracket equations is considered within the context of the Heisenberg uncertainty principle, whose philosophy is shown to have been derived by fortuitous consideration of first order commutators only.

Finally in Section 4, an example of the canonical quantum Hamilton equations is given for rotation in a plane.

2. A new equation of quantum mechanics.

The two quantum Hamilton equations developed in UFT 175 are:

$$i\hbar \frac{d}{dq} < \hat{H} > = < [\hat{H}, \hat{p}\,] > \tag{1}$$

and

$$i\hbar \frac{d}{dp} < \hat{H} > = - < [\hat{H}, \hat{q}\,] > \tag{2}$$

where $\hat{q}$ and $\hat{p}$ are the canonical operators corresponding to the canonical variables q and p of the classical Hamilton equations [11]. By using the assumption [12]:

$$\frac{d}{dq} < \hat{H} > = < \frac{d\hat{H}}{dq} > \ , \ \frac{d}{dp} < \hat{H} > = < \frac{d\hat{H}}{dp} > \tag{3}$$

these equations can be put in operator format as follows:

$$i\,\hbar\,\frac{d\hat{H}}{dq}\,\psi = [\hat{H}\,,\hat{p}\,]\,\psi \tag{4}$$

and

$$i\,\hbar\,\frac{d\hat{H}}{dp}\,\psi = -\,[\hat{H}\,,\hat{q}\,]\,\psi \tag{5}$$

where ψ is the wavefunction. In these equations $\hbar$ is the reduced Planck constant. Using the one dimensional Cartesian format:

$$\hat{q} = \hat{x} \quad , \quad \hat{p} = \hat{p} \tag{6}$$

in the position representation, the Schroedinger axiom [13, 14] is:

$$\hat{p}\,\psi = -\,i\,\hbar\,\frac{d\psi}{dx} \quad , \quad \hat{x}\,\psi = x\,\psi \tag{7}$$

and in the momentum representation:

$$\hat{x}\,\psi = \ i\,\hbar\,\frac{d\psi}{dx} \quad , \quad \hat{p}\,\psi = p\,\psi \quad . \tag{8}$$

If the hamiltonian is defined as:

$$H = \frac{p^2}{2m} + \mathrm{V}\,(x) \tag{9}$$

then

$$\frac{dH}{dx} = \frac{p^2}{2m} + \frac{dV}{dx} \tag{10}$$

because in the Hamilton dynamics x and p are independent, canonical variables. Therefore Eq. (3) is satisfied automatically. Using the result:

$$[\hat{H}, \hat{p}]\,\psi = i\hbar \frac{dV}{dx}\psi = -i\hbar F\psi \tag{11}$$

where F is force, Eq. (4) gives a new force equation of quantum mechanics:

$$\boxed{-\left(\frac{d\hat{H}}{dx}\right)\psi = F\psi} \tag{12}$$

where the eigenoperator is defined by:

$$\frac{d\hat{H}}{dx} := -\hbar^2 \frac{\partial^3}{\partial x^3} + \frac{dV(x)}{dx} \quad . \tag{13}$$

In the classical limit, the correspondence principle of quantum mechanics means that Eq. (12) becomes one of Hamilton equations:

$$F = \frac{dp}{dt} = -\left(\frac{dH}{dx}\right) \quad . \tag{14}$$

In the momentum representation Eq. (5) gives a second fundamental equation of quantum mechanics:

$$\boxed{\left(\frac{d\hat{H}}{dp}\right)\psi = v\,\psi} \tag{15}$$

where the eigenvalues are those of quantized velocity v. Here:

$$\frac{dH}{dp} = \frac{p}{m} \tag{16}$$

and

$$\left(\frac{d\hat{H}}{dp}\right)\psi = v\,\psi \tag{17}$$

Eq. (15) corresponds in the classical limit to the second Hamilton equation:

$$v = \frac{dx}{dt} = \frac{dH}{dp} \quad . \tag{18}$$

The general, or canonical, formulation of Eqs. (12) and (15) is as follows:

$$-\left(\frac{d\hat{H}}{dq}\right)\psi = F\,\psi \tag{19}$$

and

$$(\frac{d\hat{H}}{dp}) \, \psi = v \, \psi \tag{20}$$

which reduce to the canonical Hamilton equations [15]:

$$- \frac{dH}{dq} = \frac{dp}{dt} \tag{21}$$

and

$$\frac{dH}{dp} = \frac{dq}{dt} \quad . \tag{22}$$

The rotational equivalent of Eq. (4) is:

$$i \, ħ \, (\frac{d\hat{H}}{d\varphi}) \, \psi = [\hat{H} , \hat{J}_Z] \, \psi \tag{23}$$

in which the canonical variables are:

$$q = \varphi \quad , \quad \hat{p} = \hat{J}_Z \quad . \tag{24}$$

For rotational problems in the quantum mechanics of atoms and molecules, $\hat{H}$ commutes with $\hat{J}_Z$ [12 -14] , so

$$[\hat{H} , \hat{J}_Z] \, \psi = 0 \tag{25}$$

in which case:

$$(\frac{d\hat{H}}{d\varphi}) \, \psi = 0 \quad . \tag{26}$$

In order for $d\hat{H} \,/\, d\varphi$ to be non-zero, there must be a φ dependent potential energy in the hamiltonian:

$$H = \frac{J^2}{2\, I} + V\,(\varphi) \tag{27}$$

so the hamiltonian operator must be:

$$\hat{H} = - \frac{ħ^2}{2\, I} \, \hat{\Lambda}^2 + V\,(\varphi) \tag{28}$$

where $\hat{\Lambda}^2$ is the lagrangian operator [12-14]. In this case:

$$\frac{d\hat{H}}{d\varphi} = -\frac{\hbar^2}{2\,I}\hat{\Lambda}^2 + \frac{dV}{d\varphi} \tag{29}$$

and Eq. (23) gives the torque equation of quantum mechanics:

$$-\left(\frac{d\hat{H}}{d\varphi}\right)\psi = T\mathrm{q}\,\psi = -\left(\frac{dV}{d\varphi}\right)\psi \tag{30}$$

where Tq are eigenvalues of torque.

3. Higher order quantum Hamilton equations.

The higher order quantum Hamilton equations are built up straightforwardly from commutator algebra [13–15]. For example:

$$[\hat{A}\,,\hat{p}\hat{p}\,] = [\hat{A}\,,\hat{p}\,]\,\hat{p} + \hat{p}\,[\hat{A}\,,\hat{p}\,] \tag{31}$$

so the relevant quantum Hamilton equation is:

$$[\hat{A}\,,\hat{p}\hat{p}\,]\,\psi = i\,\hbar\left(\frac{d\hat{A}}{dx}\hat{p} + \hat{p}\frac{d\hat{A}}{dx}\right)\psi \quad . \tag{32}$$

If the operator $\hat{A}$ is for example:

$$\hat{A} = \hat{x}\hat{x} \tag{33}$$

then:

$$[\hat{x}\hat{x}, \hat{p}\hat{p}\,]\,\psi = i\,\hbar\left(\frac{d\hat{x}\hat{x}}{dx}\hat{p} + \hat{p}\frac{d\hat{x}\hat{x}}{dx}\right)\psi \quad . \tag{34}$$

where:

$$\frac{d\hat{x}\hat{x}}{dx}\psi = 2x\,\psi \quad . \tag{35}$$

So as in UFT 175:

$$[\hat{x}\hat{x}, \hat{p}\hat{p}\,]\,\psi = 2\,i\,\hbar\,\{\hat{x}\,,\hat{p}\,\}\,\psi \tag{36}$$

Q. E. D. In the position representation:

$$\hat{p}\,\psi = -\,i\,\hbar\frac{d\psi}{dx} \tag{37}$$

and

$$\{\hat{x}\ ,\hat{p}\ \}\ \psi\ =\hat{x}\ (\hat{p}\ \psi\ )+\hat{p}\ (\hat{x}\ \psi\ ) \tag{38}$$

so the anticommutator of $\hat{x}$ and $\hat{p}$ is:

$$\{\hat{x}\ ,\hat{p}\ \}\ \psi\ =-\ i\ \hbar\ (\ x\frac{d\psi}{dx}+\frac{d}{dx}\left(x\ \psi\right))=-\ i\ \hbar\ (\psi+2x\frac{d\psi}{dx}\ )\ . \tag{39}$$

Therefore:

$$[\hat{x}\hat{x},\hat{p}\hat{p}\ ]\ \psi=2\ \hbar^2\ \psi+4\ ix\hat{p}\ \psi \tag{40}$$

as in UFT 175.

Let $\hat{A}$ and $\hat{B}$ be two hermitian operators of quantum mechanics [13–15], then as first shown by Dirac [12]:

$$\frac{<[\hat{A},\hat{B}]>}{i\ \hbar}\ \xrightarrow[\hbar\rightarrow 0]{}\ (\ A,B\ ) \tag{41}$$

where the Poisson bracket of classical mechanics is:

$$(\ A,B\ )\ =\frac{\partial A}{\partial x}\ \frac{\partial B}{\partial p}-\frac{\partial B}{\partial x}\frac{\partial A}{\partial p}\quad . \tag{42}$$

Therefore the quantum Hamilton equations reduce as follows to the Poisson brackets:

$$\frac{d<\hat{H}>}{dx}\ \xrightarrow[\hbar\rightarrow 0]{}\ (\ H,p\ ) \tag{43}$$

$$\frac{d<\hat{H}>}{dp}\ \xrightarrow[\hbar\rightarrow 0]{}-(\ H,x\ ) \tag{43a}$$

In the classical limit:

$$\frac{dH}{dx}=(H,p)\quad , \tag{44}$$

$$\frac{dH}{dp}=-(H,x)\quad , \tag{45}$$

Now use the equations [15] of classical dynamics:

$$\frac{dp}{dt}=(p\ ,H\ )\quad ,\quad \frac{dx}{dt}=(x\ ,H\ ) \tag{46}$$

to recover the Hamilton equations self consistently:

$$\frac{dH}{dp} = \frac{dx}{dt} \quad , \quad \frac{dH}{dx} = - \frac{dp}{dt} \tag{47}$$

Q.E.D.

As shown in UFT 175 the expectation value $<[\hat{x}\hat{x},\ \hat{p}\hat{p}]>$ is zero for all the wavefunctions of the harmonic oscillator and non-zero for all the wavefunctions of the H atom. The Copenhagen interpretation of quantum mechanics is therefore refuted because if the expectation value is zero, $\hat{x}\hat{x}$ and $\hat{p}\hat{p}$ are simultaneously knowable to any precision. If the expectation value is non-zero either $\hat{x}\hat{x}$ or $\hat{p}\hat{p}$ can be absolutely unknowable, i.e. indeterminate. These claims cannot be true at the same time for the same operators from the same equation. The Copenhagen interpretation is therefore reduced to absurdity, Q.E.D.

The whole of the Copenhagen school´s claims was based on:

$$[\hat{x}\ ,\hat{p}\ ]\ \psi = i\,\hbar\ \psi \tag{48}$$

with the property:

$$\frac{<[\hat{x},\hat{p}]>}{i\,\hbar} \xrightarrow[\hbar \to 0]{} 1 = (\, x, p\,) \tag{49}$$

The Poisson bracket happens to be a constant, unity. Higher order commutators however give results such as:

$$\frac{<[\hat{x}\hat{x},\hat{p}\hat{p}]>}{i\,\hbar} \xrightarrow[\hbar \to 0]{} (x^2, p^2\ ) \tag{50}$$

which can be zero or non zero according to which wavefunction is considered. In both cases the general rule is:

$$\frac{<[\hat{A},\hat{B}]>}{i\,\hbar} \xrightarrow[\hbar \to 0]{} (\ A, B\ ) \quad . \tag{51}$$

The Copenhagen interpretation was derived wholly from:

$$(\ A, B\ ) = 1 \tag{52}$$

in which case:

$$\delta\ \hat{A}\ \ \delta\ \hat{B} \geq \frac{1}{2}\ |< \hat{C} >| \tag{53}$$

where $\delta\ \hat{A}$ and $\delta\ \hat{B}$ are root mean square deviations from the mean [12–14]. If the expectation value:

$$< \hat{C} > = \frac{1}{i} < [\hat{A}, \hat{B}] > \tag{54}$$

is not constant, the Copenhagen interpretation makes no sense as discussed already.

Finally in this section the derivation of the result (40) is cross checked with the commutator algebra [12, 13]:

$$[\hat{A}\hat{B}\ ,\ \hat{C}\hat{D}\ ]\ \psi = ([\hat{A}\hat{B}\ ,\ \hat{C}\ ]\ \hat{D} + \hat{C}\ [\hat{A}\hat{B}\ ,\ \hat{D}\ ])\ \psi \tag{55}$$

where:

$$[\hat{A}, \hat{B}\hat{C}\ ]\ \psi = (\ [\hat{A}, \hat{B}]\ \hat{C} + \hat{B}\ [\hat{A}, \hat{C}\ ])\ \psi \tag{56}$$

Thus:

$$[\hat{C}\hat{D}\ , \hat{A}\hat{B}\ ]\ \psi = (\ [\hat{C}, \hat{A}\ ]\ \hat{B}\hat{D} + \hat{A}\ [\hat{C}, \hat{B}\ ]\ \hat{D} + \hat{C}\ [\hat{D}, \hat{A}\ ]\ \hat{B} + \hat{C}\hat{A}\ [\hat{D}, \hat{B}\ ]\ )\ \psi \tag{57}$$

As for all commutator equations, this is true in all representations such as the position and momentum representations. Thus:

$$[\hat{x}\hat{x}\ , \hat{p}p\ ]\ \psi = (\ [\hat{x}, \hat{p}\ ]\ \hat{p}\hat{x} + \hat{p}\ [\hat{x}, \hat{p}\ ]\ \hat{x} + \hat{x}\ [\hat{x}, \hat{p}\ ]\ \hat{p} + \hat{x}\hat{p}\ [\hat{x}, \hat{p}\ ]\ )\ \psi \tag{58}$$

Now use:

$$[\hat{x}, \hat{p}\ ] = i\ \hbar \tag{59}$$

to find:

$$[\hat{x}\hat{x}\ , \hat{p}p\ ]\ \psi = (i\ \hbar\ \hat{p}\hat{x} + i\ \hbar\ \hat{p}\hat{x} + i\ \hbar\hat{x}\ \hat{p} + i\ \hbar\hat{x}\ \hat{p}\ )\ \psi = 2\ i\ \hbar\ [\hat{x}, \hat{p}\ ]\ \psi \tag{60}$$

Q. E. D.

4. Quantum Hamilton equations for rotation in a plane.

Consider the canonical quantum Hamilton equation:

$$i\ \hbar \frac{d\hat{H}}{dq}\ \psi = \ [\hat{H}\ , \hat{p}\ ]\ \psi \tag{61}$$

For rotation in a plane in the position representation:

$$\frac{d}{dq} = \frac{d}{d\varphi} \quad , \quad \hat{p}_\Psi = \hat{J} \tag{62}$$

where $\hat{J}$ is the angular momentum operator [12–14]. Therefore the quantum Hamilton equation for rotation in a plane is:

$$i\hbar \frac{d\hat{H}}{d\varphi}\,\psi = [\hat{H}\,,\hat{J}\,]\,\psi \tag{63}$$

corresponding to the classical Hamilton equation:

$$\frac{dH}{d\varphi} = -\frac{dJ}{dt} \quad . \tag{64}$$

The classical hamiltonian [15] is:

$$H = \frac{1}{2}\,mr^2 \left(\frac{d\varphi}{dt}\right)^2 \tag{65}$$

and the angular momentum is:

$$J = mr^2 \frac{d\varphi}{dt} \tag{66}$$

If the angular momentum commutes with the hamiltonian then it is a constant of motion [12–15]:

$$\frac{dJ}{dt} = 0 \quad . \tag{67}$$

Therefore:

$$\frac{dH}{d\varphi} = 0 \quad . \tag{68}$$

Self consistently:

$$\frac{dH}{dJ} = \frac{dH}{d\dot{\varphi}}\,\frac{d\dot{\varphi}}{dJ} = \frac{mr^2}{mr^2}\,\frac{d\varphi}{dt} = \frac{d\varphi}{dt} \quad . \tag{69}$$

The origin of the quantum Hamilton equation is the tautology:

$$i\hbar \frac{<d\hat{\varphi}>}{d\varphi}\,\psi = <[\hat{\varphi}\,,\hat{J}\,]> \tag{70}$$

which is the equivalent for planar rotational motion of the tautology developed in UFT 175:

$$i\hbar \frac{<d\hat{x}>}{dx}\,\psi = <[\hat{x}\,,\hat{p}\,]> \tag{71}$$

In the momentum representation the relevant tautology is:

$$i\,\hbar\,\frac{<d\hat{J}>}{dJ}\,\psi = -<[\hat{J},\hat{\varphi}\,]> \tag{72}$$

so:

$$i\,\hbar\,\frac{<d\hat{H}>}{dJ}\,\psi = -<[\hat{H},\hat{\varphi}\,]> \tag{73}$$

which corresponds to the classical Hamilton equation for planar rotational motion:

$$\frac{dH}{dJ} = \frac{d\varphi}{dt} \quad . \tag{74}$$

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