Non-Linearity in the ECE Equations of Electromagnetism - Part I

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Abstract

The equations of the Einstein Cartan Evans field model, for a Minkowski frame of reference, have been modified to include non-linear terms in the field equations, derived from the first Bianchi identity of Cartan geometry. It is shown that in this case the non-linear field intensity definitions provided by the Cartan identity for torsion, the first structure equation, behave in a non-linear fashion also. The full system of equations is provided for the ECE non-linear electromagnetic theory.

1. Introduction

Physicists (and engineers) often take coordinate systems, shift the origins, change the length scales, and use all sorts of modifying transformations to distort mathematical fields into something suiting them at the time. These modifying functions or operations are termed gauge transformations and fields that are invariant to these operations are terms gauge invariant [1]. Classical electromagnetism has many examples of this. The ECE field theory was thought to not be gauge invariant because of its inherent link to an absolute origin or zero point for the tetrad used to define the fields. This info was mainly derived from the ECE wave equation [2]. It will be shown that such is not the case for the field equations, that in fact, the first structural equation of Cartan, the basis for relating field intensity functions such as electric and magnetic field strengths for example, to field potentials is quite amenable to re-gauging. After that we derive a non-linear form of the field equations by re-defining the ECE charge and current densities.

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2. Gauge transformation

Let us first look at the so-called zero point for a Cartan field. The torsion tensor $T^{a}{}_{\mu\nu}$ of Cartan geometry is given by the first Cartan structure equation [2]

$$T^{a}{}_{\mu\nu} = \partial_\mu q^{a}{}_{\nu} - \partial_\nu q^{a}{}_{\mu} + \omega^{a}{}_{\mu b} q^{b}{}_{\nu} - \omega^{a}{}_{\nu b} q^{b}{}_{\mu} \qquad (1)$$

with spin connections $\omega^{a}{}_{\mu b}$. The Cartan tetrad q^{a}_{μ} maps one space-time referenced by the index "μ" to another referenced by the index "a". Often one spacetime is taken as flat (Minkowski typically), and the other is a general space-time. The second space-time can be without an orthonormal basis, etc. Note that existing ECE electromagnetic theory was developed using a cyclic coordinate system [2]. We will assume that the space-time of the observer is a more computationally friendly Minkowski space-time, given by the μ, ν indices.

There is a non-trivial state for the Cartan tetrad which exists when the torsion is zero (we will call it the ground state). It is defined by the same spin connection $\omega^{a}{}_{\mu b}$ and boundary conditions as the torsion state but is free of sources, sinks, or otherwise material influences.

This state, indexed by the superscript (gnd) is given by the following

$$\partial_\mu q^{a(gnd)}_{\ \nu} - \partial_\nu q^{a(gnd)}_{\ \mu} + \omega^{a}{}_{\mu b} q^{b\,(gnd)}_{\ \nu} - \omega^{a}{}_{\nu b} q^{b\,(gnd)}_{\ \mu} = 0 \qquad (2)$$

Equation (1) can then be re-written, by subtracting equation (2) and defining a new tetrad, one measured from a non-zero datum;

$$Q^{a}{}_{\mu} = q^{a}_{\mu} - q^{a(gnd)}_{\ \mu} \qquad (3)$$

to get

$$T^{a}{}_{\mu\nu} = \partial_\mu Q^{a}{}_{\nu} - \partial_\nu Q^{a}{}_{\mu} + \omega^{a}{}_{\mu b} Q^{b}{}_{\nu} - \omega^{a}{}_{\nu b} Q^{b}{}_{\mu}. \qquad (4)$$

If one lets the vector potential in four dimension space be proportional to $Q^{a}{}_{\nu}$ [2], it is apparent that the vacuum state has been removed from consideration by a simple shift of the zero point. It is interesting to note, that virtually all field measurements, especially evident in electromagnetism, are made from the “ground” state, which is taken to be zero. It has been shown [3] that the vacuum state given by ECE electromagnetism, is not a simple zero state, but is rich in longitudinal wave functions in potential.

A second gauge transformation, as shown in what follows, allows the non-linearities of equations (1) or (4) to be eliminated. The fundamental theorem of forms [4], which leads to the development of geometries called Rahm cohomologies of which the Minkowski space-time is an example, states that if the exterior derivative of a form is zero, then the form is exact. This is the Poincare Lemma. In the notation of differential geometry, the exterior derivative of a form A is closed when

$$dA = 0 \tag{5}$$

By the Poincare Lemma, the form is then exact, given by

$$d(dA) = 0 \tag{6}$$

or

$$B = dA \tag{7}$$

where B is a derived field quantity. This is essence eliminates the non-linear terms from equation (1).

In three dimensional Cartesian geometry, this is equivalent to

if $\quad \underline{\nabla} \cdot \boldsymbol{A} = \boldsymbol{0} \quad$ then $\quad \boldsymbol{B} = \underline{\nabla} \times \boldsymbol{A} \quad$ (8)

and

if $\quad \underline{\nabla} \times \boldsymbol{A} = 0 \quad$ then $\quad \boldsymbol{A} = \underline{\nabla}\phi \quad$ (9)

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors and ϕ is a scalar.

For those readers not experienced in Cartan differential geometry, let us look at the vector formulation of Cartan's first structural equation as developed for the electromagnetic equations of ECE theory. The torsion tensor expressed in three dimension vector format becomes [5]

$$\boldsymbol{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \boldsymbol{A^a}}{\partial t} - \omega^a_{0b}\boldsymbol{A}^b + \boldsymbol{\omega}^a_b\phi^b \; , \tag{10}$$

$$\boldsymbol{B}^a = \underline{\nabla} \times \boldsymbol{A}^a - \boldsymbol{\omega}^a_b \times \boldsymbol{A}^b \, . \tag{11}$$

The superscript "a" is the polarization of the electromagnetic field, with the standard model having this as a single value hence not shown. $\boldsymbol{A^a}$ and ϕ^a are the vector and scalar potentials for each polarization. $\boldsymbol{E}^a$ and $\boldsymbol{B}^a$ are the electric and induction field intensities and ω^a_{0b} and $\boldsymbol{\omega}^a_b$ are the scalar and vector spin connections, respectively. The Faraday and Gauss law are derived from the Bianchi identity which is in vector form:

$$\underline{\nabla} \times \boldsymbol{E}^a + \frac{\partial \boldsymbol{B}^a}{\partial t} = 0 \, , \tag{12}$$

$$\underline{\nabla} \cdot \boldsymbol{B}^a = 0 \, . \tag{13}$$

At first glance, the appearance of the potentials with absolute terms in (10) and (11), as opposed to that of derivatives in space and time as in the standard formulation, would suggest that these definitions cannot be re-gauged. However, the application of Gauss's Law given by equation (13) to the field intensity definition of equation (11) shows that

$$\underline{\nabla} \cdot (\boldsymbol{\omega}^a_b \times \boldsymbol{A}^b) = \boldsymbol{0} \tag{14}$$

and using well know results of vector calculus, equation (8), we have

$$\boldsymbol{\omega}^a_b \times \boldsymbol{A}^b = \underline{\nabla} \times \boldsymbol{F}^a \tag{15}$$

with a new field $\boldsymbol{F}^a$**.** Substitution of this, and equation (10) to Faraday's Law of equation (12) gives

$$\underline{\nabla} \times \left(-\underline{\nabla}\phi^a - \frac{\partial \boldsymbol{A^a}}{\partial t} - \omega^a_{0b}\boldsymbol{A}^b + \boldsymbol{\omega}^a_b \phi^b \right) + \frac{\partial\left(\underline{\nabla} \times \boldsymbol{A}^a - \underline{\nabla} \times \boldsymbol{F}^a\right)}{\partial t} = 0$$

which simplifies to

$$\underline{\nabla} \times \left(-\omega^a_{0b} \boldsymbol{A}^b + \boldsymbol{\omega^a_b} \phi^b - \frac{\partial \boldsymbol{F}^a}{\partial t} \right) = 0 \quad . \tag{16}$$

Again using well known vector calculus concepts given by equation (9), we can write

$$-\omega^a_{0b} \boldsymbol{A}^b + \boldsymbol{\omega^a_b} \phi^b - \frac{\partial \boldsymbol{F}^a}{\partial t} = \underline{\nabla} \psi^a \tag{17}$$

by introducing a new scalar potential ψ^a. If we change variables and write

$$\Phi^a = \phi^a - \psi^a \tag{18}$$

and

$$\boldsymbol{\mathcal{A}}^a = \boldsymbol{A^a} - \boldsymbol{F}^a \, , \tag{19}$$

one immediately has for equations (10) and (11), modified representations of the field intensities,

$$\boldsymbol{E}^a = -\underline{\nabla} \Phi^a - \frac{\partial \boldsymbol{\mathcal{A}}^a}{\partial t} \quad , \tag{20}$$

$$\boldsymbol{B}^a = \underline{\nabla} \times \boldsymbol{\mathcal{A}}^a \; . \tag{21}$$

These have the same form as the standard model definitions for the electric and magnetic fields. Through mathematically acceptable transformations, non-linear field elements inherent in the first structural equation disappear when expressed using conventional vector mathematics, irrespective of the degree of polarization being considered if the right hand side of equations (12) and (13) are zero.

3. Non-Linear Field Equations

Next we will show how non-linear terms can be found in the ECE field equations, derived from the first Bianchi identify of Cartan geometry. In indicial form, this identity is [2]

$$\partial_\mu T^{a\mu\nu} + \omega^a{}_{\mu b} T^{b\mu\nu} = R_\mu{}^{a\mu\nu} \, , \tag{22}$$

$R_{\mu}^{\ a\sigma\nu}$ is the Cartan curvature tensor.

This and its equivalent Hodge dual equation [2]

$$\partial_{\mu}\tilde{T}^{a\mu\nu} + \omega^{a}_{\ \mu b}\tilde{T}^{b\mu\nu} = \tilde{R}_{\mu}^{\ a\mu\nu} \tag{23}$$

define the ECE field equations.

The inhomogeneous current $j_I^{a\nu}$ has been until now, defined as

$$j_I^{a\nu} = R_{\mu}^{\ a\mu\nu} - \omega^{a}_{\ \mu b}T^{a\mu\nu} \tag{24}$$

resulting in a set of linear field equations

$$\partial_{\mu}T^{a\mu\nu} = j_I^{a\nu}\ . \tag{25}$$

The corresponding homogeneous field equations, a result of equation (23), are

$$\partial_{\mu}\tilde{T}^{a\mu\nu} = \tilde{R}_{\mu}^{\ a\mu\nu} - \omega^{a}_{\ \mu b}\tilde{T}^{a\mu\nu} = j_H^{a\nu} \approx 0 \tag{26}$$

where $j_H^{a\nu}$ is the homogeneous current, given experimentally to be zero or immeasurably close to it. For a single polarization these are the field equations of the standard model, and are completely linear.

If we re-define the currents of equations (25) and (26) by

$$j_I^{a\nu} = R_{\mu}^{\ a\mu\nu} \tag{27}$$

and

$$j_H^{a\nu} = \tilde{R}_{\mu}^{\ a\mu\nu} \approx 0\ , \tag{28}$$

the field equations become

$$\partial_{\mu}T^{a\mu\nu} + \omega^{a}_{\ \mu b}T^{b\mu\nu} = R_{\mu}^{\ a\mu\nu} = j_I^{a\nu} \tag{29}$$

and its Hodge dual

$$\partial_{\mu}\tilde{T}^{a\mu\nu} + \omega^{a}_{\ \mu b}\tilde{T}^{b\mu\nu} = \tilde{R}_{\mu}^{\ a\mu\nu} = j_H^{a\nu} \approx 0\ . \tag{30}$$

Using methods developed and presented earlier [5], equations (29) and (30) give non-linear electromagnetic field equations as (see Appendix I)

$$\underline{\nabla}\cdot\boldsymbol{B}^a - \boldsymbol{\omega}^a_{\ b}\cdot\boldsymbol{B}^b = 0\ , \tag{31}$$

$$\underline{\nabla}\times\boldsymbol{E}^a + \frac{\partial\boldsymbol{B}^a}{\partial t} + \omega^a_{0\ b}\boldsymbol{B}^b - \boldsymbol{\omega}^a_{\ b}\times\boldsymbol{E}^b = 0\ , \tag{32}$$

$$\underline{\nabla}\cdot\boldsymbol{D}^a - \boldsymbol{\omega}^a_{\ b}\cdot\boldsymbol{D}^b = \rho^a\ , \tag{33}$$

$$\underline{\nabla}\times\boldsymbol{H}^a - \frac{\partial\boldsymbol{D}^a}{\partial t} - \omega^a_{0\ b}\boldsymbol{D}^b - \boldsymbol{\omega}^a_{\ b}\times\boldsymbol{H}^b = \boldsymbol{J}^a\ . \tag{34}$$

$\boldsymbol{D}^a$ is the electric displacement and $\boldsymbol{H}^a$ is the magnetic field intensity.

These equations are rich in non-linearities which also appear in the corresponding wave-equations. To see the wave nature of these equations in free space, one writes

$$\boldsymbol{B}^a = \mu_0\boldsymbol{H}^a\ , \tag{35}$$

$$\boldsymbol{D}^a = \varepsilon_0\boldsymbol{E}^a \tag{36}$$

where μ_0, ε_0 are the permeability and permittivity of free space.

As in traditional electromagnetic theory [6] after lengthy vector analysis (see Appendix II) we get the non-linear coupled wave equations

$$\nabla^2\boldsymbol{E}^a - \frac{1}{c^2}\frac{\partial\omega^a_{0\ b}\boldsymbol{E}^b}{\partial t} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}^a}{\partial t^2} - \underline{\nabla}\,\boldsymbol{\omega}^a_{\ b}\cdot\boldsymbol{E}^b + \underline{\nabla}\times\boldsymbol{\omega}^a_{\ b}\times\boldsymbol{E}^b - \frac{\partial}{\partial t}\boldsymbol{\omega}^a_{\ b}\times\boldsymbol{B}^b - \underline{\nabla}\times\omega^a_{0\ b}\boldsymbol{B}^b = \mu_0\frac{\partial\boldsymbol{J}^a}{\partial t} + \frac{\underline{\nabla}\rho}{\varepsilon_0}\ , \tag{37}$$

$$\nabla^2\boldsymbol{B}^a + \frac{1}{c^2}\frac{\partial(\omega^a_{0\ b}\boldsymbol{B}^b)}{\partial t} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{B}^a}{\partial t^2} - \underline{\nabla}\boldsymbol{\omega}^a_{\ b}\cdot\boldsymbol{B}^b + \underline{\nabla}\times\boldsymbol{\omega}^a_{\ b}\times\boldsymbol{B}^b + \frac{1}{c^2}\frac{\partial(\boldsymbol{\omega}^a_{\ b}\times\boldsymbol{E}^b)}{\partial t} + \frac{1}{c^2}\underline{\nabla}\times\omega^a_{0\ b}\boldsymbol{E}^b = -\mu_0\underline{\nabla}\times\boldsymbol{J}^a\ . \tag{38}$$

These two vector equations together with the definitions (10) and (11) are insufficient to give a complete solution. The ECE antisymmetry equations [5] offer additional constraints to help complete the solution. These equations for ECE electromagnetism are

$$\frac{\partial\boldsymbol{A}^a}{\partial t} - \underline{\nabla}\phi^a + \omega^a_{0b}\boldsymbol{A}^b + \boldsymbol{\omega}^a_b\phi^b = 0\ , \tag{39}$$

$$\frac{\partial A^a_i}{\partial x_j} + \frac{\partial A^a_j}{\partial x_i} + \omega^a_{ib}A^b_j + \omega^a_{jb}A^b_i = 0\ . \tag{40}$$

Equation (39) is expressed in traditional vector format. Equation (40) cannot be expressed in vector format using standard vector operators. We suggest the following "softer" alternative. From the Tetrad postulate of Cartan geometry, the connection for a given "a" is given by [2]

$$\Gamma^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\nu. \tag{41}$$

The trace of an antisymmetric matrix is always zero, in any reference frame. Applying this to the μ, ν indices, we see

$$\partial_\mu q^a{}_\mu + \omega^a{}_{\mu b} q^a{}_\mu = 0 \qquad \textit{summed over } \mu = 0,1,2,3 \qquad . \tag{42}$$

Applied to the electromagnetic portion of ECE theory, this becomes

$$\partial_\mu A^a_\mu + \omega^a_{\mu b} A^b_\mu = 0 \quad \textit{summed over } \mu = 0,1,2,3 \; . \tag{43}$$

In vector form this is

$$-\underline{\nabla} \cdot \boldsymbol{A}^a + \frac{1}{c^2}\frac{\partial \boldsymbol{\phi}^a}{\partial t} + \omega^a_{0b}\phi^b + \boldsymbol{\omega}^a_b \cdot \boldsymbol{A}^b = 0 \tag{44}$$

Equations (39) and (44) add four equations to the system which is sufficient in principle to specify a complete solution.

We note in passing that the divergence of (39) added to the time derivative of (44) gives

$$-\nabla^2\phi^a + \frac{1}{c^2}\frac{\partial^2\phi^a}{\partial t^2} + \frac{\partial(\omega^a_{0\,b}\phi^b + \boldsymbol{\omega}^a_b \cdot \boldsymbol{A}^b)}{\partial t} + \underline{\nabla \cdot}\,(\omega^a_{0\,b}\boldsymbol{A}^b + \boldsymbol{\omega}^a_{\boldsymbol{b}} \cdot \phi^b) = 0 \tag{45}$$

4. Conclusions

The system of equations provided by (10), (11), (31-36 or 37-38), (39) and (44 or 45) constitute the full set of equations for the ECE electromagnetic theory. In part II of this paper, a reduction in the number of equations in this set will be provided, making numerical solutions much more feasible. Examples of solutions will also be provided, and experimental tests will be proposed.

Appendix I

Homogeneous Equations

We start with the Hodge Dual of the first Bianchi identity,

$$\partial_\mu \tilde{T}^{a\mu\nu} + \omega^a{}_{\mu b}\tilde{T}^{b\mu\nu} \approx 0 \;. \tag{1-1}$$

Electromagnetism is associated with this equation by defining the electric tensor as

$$F^{b\mu\nu} = A^{(0)}T^{b\mu\nu} \tag{1-2}$$

where $A^{(0)}$ is a universal constant relating geometry to the physical electromagnetic field

and

$$\tilde{F}^{a\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^a{}_{\rho\sigma} \tag{1-3}$$

is the Hodge dual of the electromagnetic tensor [5].

In matrix form, the anti-symmetric field tensor is given by [5]

$$\tilde{F}^{a\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ & 0 & \frac{E^3}{c} & \frac{-E^2}{c} \\ & & 0 & \frac{E^1}{c} \\ & & & 0 \end{bmatrix}^a \;. \tag{1-4}$$

In vector form, following [1,2,5]

$$\partial_\mu \rightarrow \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right), \tag{1-5}$$

$$\omega^a{}_{\mu b} \rightarrow \left(\frac{\omega^a_{0\,b}}{c}, -\boldsymbol{\omega}^a_b\right) \;. \tag{1-6}$$

For $\nu = 0$

$$\partial_\mu \tilde{F}^{a10} + \left(-\omega^a_{\mu\,b}\right)\tilde{F}^{b\mu 0} \approx 0 \quad \text{for} \quad \mu = 1,2,3. \tag{1-7}$$

Given the antisymmetry of $\tilde{F}^{a\mu\nu}$ this is

$$\partial_\mu \tilde{F}^{a01} + \left(-\omega^a_{\mu\, b}\right)\tilde{F}^{b0\mu} \approx 0 \quad \text{for} \quad \mu = 1,2,3. \tag{1-8}$$

Using equation (1-4) this then becomes

$$\underline{\nabla} \cdot \boldsymbol{B}^a - \boldsymbol{\omega}^a_b \cdot \boldsymbol{B}^b = 0\,. \tag{1-9}$$

For $\nu = 1$,

$$\partial_0\left(\tilde{F}^{a01}\right) + \left(\frac{\omega^a_{0\,b}}{c}\right)\left(\tilde{F}^{b01}\right) + \partial_2\left(\tilde{F}^{a21}\right) + (-\omega^a_{2\,b})\tilde{F}^{b21} + \partial_3\tilde{F}^{a31} + (-\omega^a_{3\,b})\tilde{F}^{b31} \approx 0\,,$$

using antisymmetry and (1-4), this becomes

$$\frac{1}{c}\frac{\partial}{\partial t}(-B^{a1}) + \left(\frac{\omega^a_{0\,b}}{c}\right)(-B^{b1}) + \partial_2\left(-\frac{E^{a3}}{c}\right) + (-\omega^a_{2\,b})\left(-\frac{E^{b3}}{c}\right) + \partial_3\left(-\frac{-E^{a2}}{c}\right) + (-\omega^a_{3\,b})\left(-\frac{-E^{b2}}{c}\right) \approx 0$$

or

$$\frac{1}{c}\frac{\partial}{\partial t}(B^{a1}) + \left(\frac{\omega^a_{0\,b}}{c}\right)(B^{a1}) + \partial_2\left(\frac{E^{a3}}{c}\right) - \partial_3\left(\frac{E^{a2}}{c}\right) - (\omega^a_{2\,b})\left(\frac{E^{b3}}{c}\right) + (\omega^a_{3\,b})\left(\frac{E^{b2}}{c}\right) \approx 0\,.$$

This generalizes to

$$\underline{\nabla} \times \boldsymbol{E}^a + \frac{\partial \boldsymbol{B}^a}{\partial t} + \omega^a_{0\,b}\boldsymbol{B}^b - \boldsymbol{\omega}^{\boldsymbol{a}}_{\boldsymbol{b}} \times \boldsymbol{E}^b = 0 \tag{1-10}$$

Inhomogeneous Equations

The inhomogeneous field equation is the electromagnetic form of the first Bianchi identity,

$$\partial_\mu T^{a\sigma\nu} + \omega^a{}_{\sigma b} T^{b\sigma\nu} = j_I^{a\nu}\,. \tag{1-11}$$

Now, the electromagnetic field tensor is given by

$$G^{a\sigma\nu} = \Xi^{a\sigma\nu}{}_{\rho\lambda} F^{a\rho\lambda} \tag{1-12}$$

where $\Xi^{a\sigma\nu}{}_{\rho\lambda}$ is the four space permeability-permittivity tensor.

For an isotropic material,

$$G^{a\sigma 0} = \varepsilon F^{a\sigma 0} \quad \text{for } \nu = 0,\ \sigma = 1,2,3 \tag{1-13}$$

$$G^{a\sigma\nu} = \mu F^{a\sigma\nu} \quad \text{for } \sigma, \nu = 1,2,3. \tag{1-14}$$

We thus get

$$\partial_\mu G^{a\sigma\nu} + \omega^a{}_{\sigma b} G^{b\sigma\nu} = j_I^{a\nu} \; . \tag{1-15}$$

In matrix form

$$G^{a\mu\nu} = \begin{bmatrix} 0 & -D^1 & -D^2 & -D^3 \\ & 0 & \frac{-H^3}{c} & \frac{H^2}{c} \\ & & 0 & \frac{-H^1}{c} \\ & & & 0 \end{bmatrix}^a . \tag{1-16}$$

In addition to (1-5) and (1-6) we have

$$J^a = \left(\rho^a, \frac{\boldsymbol{J}^a}{c}\right) \tag{1-17}$$

For $\nu = 0$, equation (1-15) becomes

$$\partial_\sigma G^{a\sigma 0} + \omega^a{}_{\sigma b} G^{b\sigma 0} = j_I^{a0} \; .$$

which becomes upon using (1-16) and antisymmetry

$$\partial_\sigma (D^{a\sigma 0}) + (-\omega^a_{\sigma b})(D^{b\sigma 0}) = \rho^a.$$

Generalizing, this is

$$\underline{\nabla} \cdot \boldsymbol{D}^a - \boldsymbol{\omega}^{\boldsymbol{a}}_{\boldsymbol{b}} \cdot \boldsymbol{D}^b = \rho^a \; . \tag{1-18}$$

For $\nu = 1$, equation (1-15) becomes

$$\partial_0 G^{a01} + \omega^a{}_{0b} G^{b01} + \partial_2 G^{a21} + (-\omega^a{}_{2b}) G^{b21} + \partial_3 G^{a31} + (-\omega^a{}_{3b}) G^{b31} = j_I^{a1}$$

which gives

$$\frac{1}{c}\frac{\partial}{\partial t}(-D^{a1}) + \left(\frac{\omega^a_{0\,b}}{c}\right)(-D^{b1}) + \partial_2\left(\frac{H^{a3}}{c}\right) + (-\omega^a_{2\,b})\left(\frac{H^{b3}}{c}\right) + \partial_3\left(\frac{-H^{a2}}{c}\right) + (-\omega^a_{3\,b})\left(-\frac{-H^{b2}}{c}\right) \approx \frac{J^{a1}}{c}$$

or

$$\frac{1}{c}\frac{\partial}{\partial t}(D^{a1}) + \left(\frac{\omega^{a}_{0\ b}}{c}\right)(D^{b1}) - \partial_2\left(\frac{H^{a3}}{c}\right) + \partial_3\left(\frac{H^{a2}}{c}\right) + (\omega^{a}_{2\ b})\left(\frac{H^{b3}}{c}\right) - (\omega^{a}_{3\ b})\left(\frac{H^{b2}}{c}\right) \approx \frac{J^{a1}}{c} \; .$$

This generalizes to

$$\underline{\nabla} \times \boldsymbol{H}^{a} - \frac{\partial \boldsymbol{D}^{a}}{\partial t} - \omega^{a}_{0\ b}\boldsymbol{D}^{b} - \boldsymbol{\omega}^{\boldsymbol{a}}_{\boldsymbol{b}} \times \boldsymbol{H}^{b} = \boldsymbol{J}^{a} \; . \tag{1-19}$$

Appendix II

From equations (31)–(36)

$$\underline{\nabla} \cdot \boldsymbol{B}^a - \boldsymbol{\omega}^a_{\ b} \cdot \boldsymbol{B}^b = 0 \ , \qquad (2\text{-}1)$$

$$\underline{\nabla} \times \boldsymbol{E}^a + \frac{\partial \boldsymbol{B}^a}{\partial t} + \omega^a_{0\ b}\boldsymbol{B}^b - \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{E}^b = 0 \ , \qquad (2\text{-}2)$$

$$\underline{\nabla} \cdot \boldsymbol{E}^a - \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \cdot \boldsymbol{E}^a = \frac{\rho^a}{\varepsilon_0} \ , \qquad (2\text{-}3)$$

$$\underline{\nabla} \times \boldsymbol{B}^a - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^a}{\partial t} - \frac{1}{c^2}\omega^a_{0\ b}\boldsymbol{E}^b - \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{B}^b = \mu_0 \boldsymbol{J}^a \ . \qquad (2\text{-}4)$$

Taking the Curl of ((2-2)

$$\underline{\nabla} \times \left(\underline{\nabla} \times \boldsymbol{E}^a\right) + \underline{\nabla} \times \frac{\partial \boldsymbol{B}^a}{\partial t} + \underline{\nabla} \times \omega^a_{0\ b}\boldsymbol{B}^b - \underline{\nabla} \times \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{E}^b = 0 \ .$$

This expands, using the well known vector identity for the double curl,

$$\underline{\nabla} \times \left(\underline{\nabla} \times \boldsymbol{A}\right) = -\nabla^2 A + \underline{\nabla}\,\underline{\nabla} \cdot \boldsymbol{A}$$

and the time derivative of (2-4),

$$-\nabla^2 \boldsymbol{E}^a + \underline{\nabla}\,\underline{\nabla} \cdot \boldsymbol{E}^a + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \boldsymbol{E}^a}{\partial t} + \frac{1}{c^2}\omega^a_{0\ b}\boldsymbol{E}^b + \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{B}^b + \mu_0 \boldsymbol{J}^a\right) + \underline{\nabla} \times \omega^a_{0\ b}\boldsymbol{B}^b - \underline{\nabla} \times \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{E}^b = 0 \ .$$

Using equation (2-3) for $\underline{\nabla} \cdot \boldsymbol{E}^a$ this becomes

$$-\nabla^2 \boldsymbol{E}^a + \underline{\nabla}\left(\boldsymbol{\omega}^a_{\ b} \cdot \boldsymbol{E}^b + \frac{\rho^a}{\varepsilon_0}\right) + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \boldsymbol{E}^a}{\partial t} + \frac{1}{c^2}\omega^a_{0\ b}\boldsymbol{E}^b + \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{B}^b + \mu_0 \boldsymbol{J}^a\right) + \underline{\nabla} \times \omega^a_{0\ b}\boldsymbol{B}^b - \underline{\nabla} \times \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{E}^b = 0 \ .$$

Rearranging terms

$$\nabla^2 \boldsymbol{E}^a - \frac{1}{c^2}\frac{\partial \omega^a_{0\ b}\boldsymbol{E}^b}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}^a}{\partial t^2} - \underline{\nabla}\,\boldsymbol{\omega}^a_{\ b} \cdot \boldsymbol{E}^b + \underline{\nabla} \times \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{E}^b - \frac{\partial}{\partial t}\boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{B}^b - \underline{\nabla} \times \omega^a_{0\ b}\boldsymbol{B}^b = \mu_0 \frac{\partial \boldsymbol{J}^a}{\partial t} + \frac{\underline{\nabla}\rho}{\varepsilon_0} \ . \qquad (2\text{-}5)$$

Similarly, taking the Curl of (2-4)

$$\underline{\nabla} \times \underline{\nabla} \times \boldsymbol{B}^a - \frac{1}{c^2}\frac{\partial \underline{\nabla} \times \boldsymbol{E}^a}{\partial t} - \frac{1}{c^2}\underline{\nabla} \times \omega^a_{0\ b}\boldsymbol{E}^b - \underline{\nabla} \times \boldsymbol{\omega}^{\boldsymbol{a}}_{\ \boldsymbol{b}} \times \boldsymbol{B}^b = \mu_0 \underline{\nabla} \times \boldsymbol{J}^a \ . \qquad (2\text{-}6)$$

Expanding the double curl and using (2-2) for $\underline{\nabla} \times \boldsymbol{E}^a$ gives

$$-\nabla^2\boldsymbol{B}^a+\underline{\nabla}\,\underline{\nabla}\cdot\boldsymbol{B}^a-\frac{1}{c^2}\frac{\partial\left(-\frac{\partial\boldsymbol{B}^a}{\partial t}-\omega_{0\ b}^{\ a}\boldsymbol{B}^b+\boldsymbol{\omega}_{\boldsymbol{b}}^{\boldsymbol{a}}\times\boldsymbol{E}^b\right)}{\partial t}-\frac{1}{c^2}\underline{\nabla}\times\omega_{0\ b}^{\ a}\boldsymbol{E}^b-\underline{\nabla}\times\boldsymbol{\omega}_{\boldsymbol{b}}^{\boldsymbol{a}}\times\boldsymbol{B}^b=\mu_0\underline{\nabla}\times\boldsymbol{J}^a\,.$$

Using (2-1)

$$-\nabla^2\boldsymbol{B}^a+\underline{\nabla}\boldsymbol{\omega}_{\ b}^{a}\cdot\boldsymbol{B}^b+\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}^a}{\partial t^2}+\frac{1}{c^2}\frac{\partial\left(\omega_{0\ b}^{\ a}\boldsymbol{B}^b-\boldsymbol{\omega}_{\boldsymbol{b}}^{\boldsymbol{a}}\times\boldsymbol{E}^b\right)}{\partial t}-\frac{1}{c^2}\underline{\nabla}\times\omega_{0\ b}^{\ a}\boldsymbol{E}^b-\underline{\nabla}\times\boldsymbol{\omega}_{\boldsymbol{b}}^{\boldsymbol{a}}\times\boldsymbol{B}^b=\mu_0\underline{\nabla}\times\boldsymbol{J}^a$$

or upon rearranging

$$\nabla^2\boldsymbol{B}^a+\frac{1}{c^2}\frac{\partial\left(\omega_{0\ b}^{\ a}\boldsymbol{B}^b\right)}{\partial t}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}^a}{\partial t^2}-\underline{\nabla}\boldsymbol{\omega}_{\ b}^{a}\cdot\boldsymbol{B}^b+\frac{1}{c^2}\frac{\partial\left(\boldsymbol{\omega}_{\boldsymbol{b}}^{\boldsymbol{a}}\times\boldsymbol{E}^b\right)}{\partial t}+\frac{1}{c^2}\underline{\nabla}\times\omega_{0\ b}^{\ a}\boldsymbol{E}^b+\underline{\nabla}\times\boldsymbol{\omega}_{\boldsymbol{b}}^{\boldsymbol{a}}\times\boldsymbol{B}^b=-\mu_0\underline{\nabla}\times\boldsymbol{J}^a\,. \quad (2\text{-}7)$$

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