The Equivalence Theorem of Cartan Geometry and General Relativity

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Abstract

The consideration of active and passive rotations in a plane is developed into the equivalence principle of Cartan geometry and general relativity, valid for all types of motion in any mathematical space in any number of dimensions. The equivalence theorem states that the ordinary four derivative of any vector is equal to the product of that vector with the spin connection provided that these are the two components of the covariant derivative. The theorem is valid for any self consistent geometry, is illustrated with planar orbits of any kind and a new general relativity developed in which all orbits are described by spin connection components.

Keywords: ECE theory, equivalence theorem of Cartan geometry, general relativity, orbits.

1. Introduction

In recent papers of this series {1 – 10} it has been demonstrated that the Einsteinian general relativity is incorrect and obsolete, and the first attempts have been made to develop a new theory of general relativity. There are many errors in the old type of general relativity, some of which have been known for nearly a century. In order to begin to develop a valid relativity theory, the basic geometry must be correct and self consistent. Such a geometry has been available since the early nineteen twenties, and is due to Cartan and coworkers {11}. It is developed from two structure equations that define the torsion and curvature. These two quantities are linked by an identity that correctly incorporates torsion. One of the major errors of the obsolete relativity was the incorrect assertion that the curvature is non-zero and the torsion is zero. In the accompanying note 199(6) on www.aias.us it is proven that this assertion is incorrect. Another major error of the obsolete relativity was the assertion that the Christoffel connection (inferred in the eighteen sixties) is symmetric in its lower two indices. This error was made in an era circa 1900 to 1920 when the torsion was unknown, but the curvature was known and first inferred by Levi-Civita and coworkers. Riemann worked in the early nineteenth century and inferred only the metric. Unfortunately Einstein inferred general relativity circa 1915, when torsion was unknown. Einstein's work was immediately and severely criticised by Schwarzschild in December 1915 {12} in a letter which is now available on the web in English translation. The torsion was inferred by Cartan and coworkers in the early twenties in Paris. Cartan and Einstein corresponded, but Einstein did not adopt the structure equations and identity of Cartan. These were first used correctly in this series of papers {1 - 10}, starting in March 2003. The work reported in this series is now known as the ECE theory of unified physics. It is accepted as the new model of physics, the standard model being riddled with errors, complexity and unworkable obscurity. The incorrect assertion of a symmetric connection resulted from the incorrect assertion that torsion is zero and curvature non-zero. The correct method of generating curvature and torsion uses {1 - 11} the operator known as the commutator of covariant derivatives. The action of this operator on any tensor in any space in any number of dimensions produces both the curvature and torsion. It is incorrect to omit the torsion because it is always produced by the commutator. The latter describes a round trip {11} or holonomy as is well known. Without the commutator there is no round trip, and the curvature and torsion cannot be defined. The commutator is antisymmetric by definition, the antisymmetry represents the round trip. The action of the commutator produces combinations of connections, connection products and derivatives of connections. The connection is always antisymmetric in its lower two indices by definition of a round trip. The torsion is a combination of two connections, the curvature is a combination of derivatives and products of connections. There is a one to one correspondence between the indices of the connection and the commutator. The connection itself is not a tensor, but the combinations known as torsion and curvature are tensors which retain their format under the general coordinate transformation. The Cartan structure equations are elegant statements which are equivalent to the action of the commutator.

The connection itself is defined by the covariant derivative of any vector in any mathematical space in any number of dimensions. The covariant derivative is made up of the sum of the ordinary derivative and a term in the connection. This sum is defined in such a way that the covariant derivative retains its format under the general coordinate transformation. The ordinary derivative does not do so {1 - 11}. In Section 2, a new theorem of Cartan geometry, or any self consistent geometry, is inferred. The theorem states that the two terms of the covariant derivative are equal. The simplest demonstration of this fact is active and passive rotation in a plane, and this is explained in Section 2. In Section 3 the new theorem is used to describe any orbit in terms of spin connection components, a new theory of general relativity valid for all observable orbits. This appears to be the simplest and most elegant way of developing the theory of general relativity.

2. Development of the Equivalence Theorem of Geometry.

Consider the rotation of the position vector defined in the plane XY:

$$\boldsymbol{r} = X\boldsymbol{i} + Y\boldsymbol{j} \tag{1}$$

In the active rotation, the position vector is rotated with fixed axes clockwise through an angle θ, producing the result:

$$\begin{aligned}\boldsymbol{r}' &= X'\boldsymbol{i} + Y'\boldsymbol{j} \\ &= (X\cos\theta + Y\sin\theta)\boldsymbol{i} + (-X\sin\theta + Y\cos\theta)\boldsymbol{j}\end{aligned} \tag{2}$$

Therefore the matrix equation:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \tag{3}$$

defines the rotation tetrad:

$$q^a_\mu = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

and example of the Cartan tetrad. In the active rotation the axes are fixed, so the Cartan spin connection is zero. In this case the Cartan torsion is defined by {1 - 11}:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu \tag{5}$$

Note that the rotation tetrad (4) is defined by:

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{bmatrix}, \tag{6}$$

$$e_r = i\cos\theta + j\sin\theta\ , \tag{7}$$

$$e_\theta = -i\sin\theta + j\cos\theta\ , \tag{8}$$

Where e_r and e_θ are the unit vectors of the cylindrical polar coordinates (r, θ) in the XY plane. The well known rotation generator {1 - 11} is inferred from the rotation tetrad, and the angular momentum operator in quantum mechanics is the rotation generator within the reduced Planck constant $\hbar$.

Now consider the passive rotation that is equivalent to the active rotation. In the passive rotation the position vector is constant but the axes are rotated anticlockwise. Write Eq. (2) as:

$$r' = X(i\cos\theta - j\sin\theta) + Y(i\sin\theta + j\cos\theta) \tag{9}$$

where X and Y in Eq. (2) are constant. The passive rotation is therefore:

$$i' = i\cos\theta - j\sin\theta \tag{10}$$

$$j' = i\sin\theta + j\cos\theta \tag{11}$$

and is a rotation of the Cartesian unit vectors i and j represented by the matrix equation:

$$\begin{bmatrix} i' \\ j' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} \tag{12}$$

Note that:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

so the inverse rotation tetrad is:

$$q_a^\mu = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{14}$$

In Cartan’s notation:

$$q_\mu^a \, q_a^\mu = 1 \tag{15}$$

For the passive rotation the components X and Y are constant, so their derivatives vanish and the passive rotation is described in Cartan geometry by the spin connection term of the covariant derivative. In this case the Cartan torsion is defined by:

$$T^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu = \omega^a_{\mu\nu} - \omega^a_{\nu\mu} \qquad (16)$$

By antisymmetry {1 - 10}:

$$T^a_{\mu\nu} = 2\partial_\mu q^a_\nu = 2\omega^a_{\mu\nu} \qquad (17)$$

and the equivalence theorem of Cartan geometry is inferred:

$$\partial_\mu q^a_\nu = \omega^a_{\mu\nu} \qquad (18)$$

The complete covariant derivative in Cartan geometry is:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \qquad (19)$$

and the new equivalence theorem of this paper shows that the two terms of the covariant derivative are the same. This result is true for any self consistent geometry in any mathematical space in any number of dimensions.

3. Application to General Relativity, Planar Orbital Theory.

A planar orbit of any kind is described by Eq. (1) in which X and Y are functions of time *t*. So for any planar orbit:

$$V^1 = X(t), \quad V^2 = Y(t) \qquad (20)$$

It follows that:

$$\partial_0 X = \omega^1_{0b} V^b = \omega^1_{01} X + \omega^1_{02} Y \quad , \qquad (21)$$

$$\partial_0 Y = \omega^2_{0b} V^b = \omega^2_{01} X + \omega^2_{02} Y \quad , \qquad (22)$$

where:

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \tag{23}$$

Here c is the speed of light in a vacuum. Transforming to cylindrical coordinates:

$$X = r\cos\theta, \qquad Y = r\sin\theta$$

(24)

In this notation:

$$(1,2) = (r, \theta) \tag{25}$$

Note both *r* and θ depend on time. So the time derivative must be worked out with the Leibniz Theorem as follows:

$$\frac{\partial}{\partial t}(r\cos\theta) = \frac{\partial r}{\partial t}\cos\theta + r\frac{\partial}{\partial t}(\cos\theta(t)) \tag{26}$$

Using the chain rule:

$$\frac{\partial f(\theta)}{\partial t} = \frac{\partial f(\theta)}{\partial \theta}\frac{\partial \theta}{\partial t} \tag{27}$$

it is found that:

$$\frac{\partial}{\partial t}(\cos\theta(t)) = -\frac{\partial \theta}{\partial t}\sin\theta \tag{28}$$

so

$$\frac{\partial}{\partial t}(r\cos\theta) = \frac{\partial r}{\partial t}\cos\theta - \frac{\partial \theta}{\partial t}\, r\sin\theta \tag{29}$$

Similarly:

$$\frac{\partial}{\partial t}(r\sin\theta) = \frac{\partial r}{\partial t}\sin\theta + \frac{\partial \theta}{\partial t}\, r\cos\theta \tag{30}$$

Comparing Eqs. (21) and (29)

$$c\,\omega^1_{\ 01} r\cos\theta + c\,\omega^1_{\ 02} r\sin\theta = \frac{\partial r}{\partial t}\cos\theta - \frac{\partial \theta}{\partial t}\, r\sin\theta \tag{31}$$

so the spin connection components are

$$\omega^1_{\ 01} = \frac{1}{c\,r}\frac{\partial r}{\partial t}\ ,$$

(32)

$$\omega^1_{\ 02} = -\frac{1}{c}\frac{\partial \theta}{\partial t}$$

(33)

Comparing Eqs. (22) and (30) gives the other two spin connection components:

$$\omega^2_{\ 01} = \frac{1}{c}\frac{\partial \theta}{\partial t} \tag{34}$$

$$\omega^2_{\ 02} = \frac{1}{c\,r}\frac{\partial r}{\partial t} \tag{35}$$

The complete spin connection for each a is therefore:

$$\omega^a_{\ 0b} = \begin{bmatrix} \omega^1_{\ 01} & \omega^1_{\ 02} \\ \omega^2_{\ 01} & \omega^2_{\ 02} \end{bmatrix} = \frac{1}{c}\begin{bmatrix} \frac{1}{r}\frac{\partial r}{\partial t} & -\frac{\partial \theta}{\partial t} \\ \frac{\partial \theta}{\partial t} & \frac{1}{r}\frac{\partial r}{\partial t} \end{bmatrix} \tag{36}$$

or:

$$r\,c\,\omega^{a}_{0b} = \begin{bmatrix} \frac{\partial r}{\partial t} & -r\frac{\partial \theta}{\partial t} \\ r\frac{\partial \theta}{\partial t} & \frac{\partial r}{\partial t} \end{bmatrix} \tag{37}$$

The orbital velocity in cylindrical coordinates has been developed in recent papers of this series and is:

$$\mathbf{v} = \mathbf{e}_r \frac{dr}{dt} + \mathbf{e}_\theta r \frac{d\theta}{dt} \tag{38}$$

It is seen that the components of the linear orbital velocity are spin connection components

$$\begin{aligned} \mathbf{v} &= r\,c\left(\omega^{1}_{01}\mathbf{e}_r - \omega^{1}_{02}\mathbf{e}_\theta\right) \\ &= r\,c\left(\omega^{2}_{02}\mathbf{e}_r + \omega^{2}_{01}\mathbf{e}_\theta\right) \end{aligned} \tag{39}$$

a result that is true for any planar orbit. This is a fully relativistic description of any orbit in terms of the connection because the orbit is described as a movement of the axes. From the rotation tetrad (4) and the spin connection components, the torsion can be evaluated using the first Cartan structure equation. The curvature can be evaluated from the connection components using the second Cartan structure equation. The field equations are given by the Cartan identity {1 - 10}. Note that only the concept of velocity has been used. In later papers the momentum and energy will be defined, together with conservation theorems of the new general relativity.

In accompanying notes 199(1) to 199(5) discussions are given of torsion driven orbital motion and torsion driven motion in a straight line; the ellipse represented as a tetrad, various representations of the tetrad of the ellipse; and notes on the Cartesian and cylindrical polar representations of the ellipse. In the solar system it is known by observation that the functional relation between *r* and θ is the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \tag{40}$$

where 2α is the right latitude, ϵ the ellipticity and *x* the precession constant. Using Eq. (40) the spin connection components may be defined for the precessing ellipse. This procedure is valid for any planar orbit in which the dependence of *r* on θ is known by observation.

Finally, note that the whole of classical dynamics can be developed in terms of spin connections using the new equivalence theorem (18), so classical and relativistic dynamics also become unified concepts. The reason is that any motion has its equivalent frame dynamics.

Acknowledgments

The British Government is thanked for a Civil List Pension, and the staff of AIAS and others for many interesting discussions. Dave Burleigh is thanked for posting, Robert Cheshire, Simon Clifford and Alex Hill for broadcasting and translation. The AIAS is established under the aegis of the Newlands Family Trust registered 2011.

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