APPLICATIONS OF ECE THEORY: THE RELATIVISTIC KINEMATICS OF ORBITS

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ABSTRACT

A relativistic theory of orbits is developed based on the Minkowski force equation, a classical limit of ECE theory. For all planar orbits the relativistic Coriolis acceleration vanishes, so the relativistic force law includes only the relativistic centripetal acceleration. The Lorentz factor is expressed in terms of the angle theta of the plane polar coordinates, allowing animations to be developed of the relativistic motion of objects in planar orbits of all kinds.

Keywords: ECE theory, relativistic kinematics of orbits, animations.

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1. INTRODUCTION

In recent papers of this series {1 - 10} the theory of planar orbits has been developed in terms of Cartan geometry, upon which ECE theory is based as is well known. The usual approach to the theory of orbits is based {11} on classical dynamics, with adjustments from Einsteinian general relativity. In the solar system these adjustments are very small, but in objects such as whirlpool galaxies the orbits of stars around the centre of the galaxy obey neither Newtonian nor Einsteinian dynamics. In preceding papers it has been shown that the basics of whirlpool galaxies can be explained very simply with non relativistic kinematics in a plane. This reasoning results in the inference that the stars move outward from the centre, move on a hyperbolic spiral trajectory, and reach a constant velocity at infinite distance r from the centre, as observed experimentally. In Section 2 we begin the exploration of an entirely new approach to relativistic orbits based on the Minkowski force equation {11}. This is a course that relativity theory could have taken in 1905, but it followed Einstein's use of Riemann geometry, a subject that became known as general relativity. Unfortunately Einstein's mathematics are now known {1 - 10} and widely accepted {12} to be riddled with errors, notably the omission of torsion, so all inferences based on Einsteinian general relativity are meaningless. This fact has actually been known for almost a hundred years and the flaws in the Einstein field equation were discovered by Schwarzschild barely a month after Einstein's publication of perihelion precession theory in 1915 {1 - 10}. It is now widely accepted that general relativity is unscientific dogma. Obviously, it fails completely to describe the universe, because it fails completely to describe whirlpool galaxies.

The Minkowski force equation is the Newton force equation with the proper time τ replacing the time in the laboratory frame t. The equation was inferred by Minkowski shortly after Einstein's introduction of the idea of relativistic momentum. In Section 2 it is shown that the Lorentz factor enters into the force equation via the proper time and this fact changes the force law needed to keep an object of mass m in orbit. The theory of Section 2 is developed with kinematics and so is a perfectly general theory. It can be reduced to Newtonian dynamics but never reduces to Einsteinian general relativity (EGR). This is now a requirement, because EGR is incorrect mathematically {1 - 10}. Newtonian dynamics is correct mathematically, but is conceptually lacking as is well known, for example Newtonian dynamics do not give the Coriolis force {11} and does not develop the centripetal and centrifugal forces self consistently. The reason for this is that Newtonian dynamics uses a static frame of reference, but a rotating frame of reference is needed for the Coriolis and centripetal forces and also for the orbital velocity {1 - 11}. In Section 2 it is shown that the space part of the Minkowski four-force {11} produces new and unexpected orbital properties that can be tested experimentally. These properties are graphed and animated in Section 3.

2. THE RELATIVISTIC FORCE LAW AND RELATIVISTIC ORBITS OF THE MINKOWSKI EQUATION.

Consider the relativistic velocity in plane polar coordinates:

$$\underline{v} = \frac{d\underline{r}}{d\tau} = \gamma \frac{d\underline{r}}{dt} \quad -(1)$$

where τ is the proper time and γ the Lorentz factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(2)$$

where c is the assumed constant speed of light in a vacuum. The relativistic acceleration is:

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) = \frac{d}{d\tau}\left(\gamma \frac{d\underline{r}}{dt}\right) = \gamma \frac{d}{dt}\left(\gamma \frac{d\underline{r}}{dt}\right) \quad -(3)$$

Using the Leibnitz Theorem:

$$\underline{a} = \gamma \left(\frac{d\gamma}{dt} \frac{d\underline{r}}{dt} + \gamma \frac{d}{dt} \left(\frac{d\underline{r}}{dt} \right) \right). \quad -(4)$$

The velocity v appearing in the Lorentz factor is defined by the infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(5)$$

where:

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2. \quad -(6)$$

Therefore:

$$c^2 d\tau^2 = (c^2 - v^2) dt^2 \quad -(7)$$

and the Lorentz factor is:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2} \right)^{-1/2}. \quad -(8)$$

In plane polar coordinates:

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2. \quad -(9)$$

Therefore:

$$v^2 = \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2. \quad -(10)$$

In plane polar coordinates {1 - 11}:

$$\underline{r} = r \underline{e}_r \quad -(11)$$

therefore the non- relativistic velocity is {1- 10}:

$$\underline{v} = \frac{d}{dt}\left(r \underline{e}_r \right) = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt}$$

$$= \frac{dr}{dt}\underline{e}_r + \omega r \underline{e}_\theta = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(12)$$

$$= \left(\frac{L_0}{m} \right)\left(\frac{1}{r}\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r} \right)\underline{e}_r \right)$$

where the unit vectors of the plane polar coordinate system are:

$$\underline{e}_r = \cos\theta \, \underline{i} + \sin\theta \, \underline{j} \quad -(13)$$

$$\underline{e}_\theta = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j} \quad -(14)$$

as discussed in immediately preceding papers of this series on www.aias.us.

The non relativistic total angular momentum is a constant of motion:

$$L_0 = mr^2 \frac{d\theta}{dt} \quad -(15)$$

where the non relativistic angular velocity is:

$$\omega = \frac{d\theta}{dt} . \quad -(16)$$

For a particle of mass m in orbit, its relativistic momentum is:

$$\underline{p} = \gamma m \frac{d\underline{r}}{dt} = m \frac{d\underline{r}}{d\tau} \quad -(17)$$

an equation which can be re arranged as follows:

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{v}{c} \right)^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2} \right) = \gamma^2 m^2 c^4 - m^2 c^4 \quad -(18)$$

giving the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(19)$$

in which

$$E = \gamma m c^2 \quad -(20)$$

is the total energy, and

$$E_0 = m c^2 \quad -(21)$$

is the rest energy.

The relativistic total angular momentum is:

$$L = m r^2 \frac{d\theta}{d\tau} = \gamma L_0 \quad -(22)$$

and is the constant of motion L_0 multiplied by the Lorentz factor γ .

It is obvious that the concept of the Minkowski force equation uses acceleration, so this type of relativity can be used to describe the force law of orbits in a much more straightforward way than EGR, which did not consider plane polar coordinates. When Einstein was asked how he would incorporate angular motions into EGR he reply that he did not know. In fact his own energy equation is easily derived from plane polar coordinates as follows. Consider the infinitesimal line element (5), and develop it as:

$$m c^2 = m c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2$$

$$= \gamma^2 m c^2 - \left(\left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2\right) = \frac{E^2}{m c^2} - \frac{p^2}{c^2} . \quad -(23)$$

So:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(24)$$

which is the Einstein energy equation, Q. E. D. Note that the relativistic linear momentum in

Eq. (23) is:

$$p^2 = m^2\left(\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right) \quad -(25)$$

which is Eq. (17), Q. E. D. The definition of relativistic acceleration is:

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) = \gamma\left(\frac{d\gamma}{dt}\frac{d\underline{r}}{dt} + \gamma\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right). \quad -(26)$$

The quantities appearing in this definition can be expressed in plane polar coordinates as follows:

$$\frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(27)$$

and

$$\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2 r}{dt^2}\underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}). \quad -(28)$$

The relativistic acceleration contains more terms than the non relativistic acceleration.

Using the chain rule {11}:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} \quad -(29)$$

where v is the velocity of the Lorentz factor defined by Eq. (10). Therefore:

$$\frac{d\gamma}{dv} = \frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \gamma^3\frac{v}{c^2} \quad -(30)$$

and in plane polar coordinates:

$$\underline{a} = \gamma^4 \frac{v}{c^2}\frac{dv}{dt}\frac{d\underline{r}}{dt} + \gamma^2\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right), \quad -(31)$$

$$= \left(\frac{d\gamma}{d\tau}\frac{dr}{dt} + \gamma^2\frac{d^2 r}{dt^2}\right)\underline{e}_r + \gamma^2\underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\gamma}{d\tau}\underline{\omega} \times \underline{r} + \gamma^2\left(\frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r\right)$$

In static Cartesian coordinates on the other hand:

$$\underline{a} = \frac{d}{d\tau}\left(\gamma \frac{d\underline{r}}{dt}\right) = \gamma \frac{d\gamma}{dt}\frac{d\underline{r}}{dt} + \gamma^2 \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) \quad -(32)$$

So:

$$\underline{a}\,(\text{Cartesian}) = \left(\gamma \frac{d\gamma}{dt}\frac{dr}{dt} + \gamma^2 \frac{d^2 r}{dt^2}\right)\underline{e}_r \quad -(33)$$

in which:

$$v = \frac{dr}{dt}, \quad \frac{d^2 r}{dt^2} = \frac{dv}{dt}, \quad \frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(34)$$

and

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} = \frac{\gamma^3 v}{c^2}\frac{dv}{dt}. \quad -(35)$$

Therefore:

$$\underline{a}\,(\text{Cartesian}) = \left(\gamma^4 \frac{v^2}{c^2} + \gamma^2\right)\frac{dv}{dt}\underline{e}_r \quad -(36)$$

in which:

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}. \quad -(37)$$

and therefore:

$$\underline{a}\,(\text{Cartesian}) = \gamma^4 \frac{d^2 r}{dt^2}\underline{e}_r \quad -(38)$$

Using Eq. (38) in Eq. (31):

$$\underline{a}\,(\text{plane polar}) = \gamma^4 \frac{d^2 r}{dt^2}\,\underline{e}_r + \gamma^2\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\gamma}{d\tau}\,\underline{\omega}\times\underline{r} + \gamma^2\left(\frac{d\underline{\omega}}{dt}\times\underline{r} + 2\,\underline{\omega}\times\frac{dr}{dt}\,\underline{e}_r\right) \quad -(39)$$

which is the expression for relativistic acceleration in plane polar coordinates.

It can be proven as follows that the relativistic Coriolis acceleration vanishes for all planar orbits. The general expression {1 - 11} for the relativistic Coriolis acceleration is:

$$\underline{a}\,(\text{Coriolis}) = \gamma^2\left(r\frac{d}{dt}\frac{d\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\underline{e}_\theta \quad -(40)$$

in which the total relativistic angular momentum is:

$$L_0 = mr^2\frac{d\theta}{dt}. \quad -(41)$$

It follows that:

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{dt}\left(\frac{L_0}{mr^2}\right) = \frac{d}{dr}\left(\frac{L_0}{mr^2}\right)\frac{dr}{d\tau} = -\frac{2L_0}{mr^3}\frac{dr}{dt} \quad -(42)$$

so:

$$\underline{a}\,(\text{Coriolis}) = \left(-\frac{2L_0}{mr^2}\frac{dr}{dt} + \frac{2L_0}{mr^2}\frac{dr}{dt}\right)\underline{e}_\theta = \underline{0} \quad -(43)$$

Q. E. D.

Therefore the relativistic acceleration for all planar orbits is

$$\underline{a} = \gamma^4\frac{d^2 r}{dt^2}\,\underline{e}_r + \gamma^2\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) + \frac{d\gamma}{d\tau}\,\underline{\omega}\times\underline{r} \quad -(44)$$

The relativistic centripetal component of this orbit is:

$$\underline{a}(\text{centripetal}) = \gamma^2 \underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\frac{L^2}{m^2 r^3} \underline{e}_r \quad -(45)$$

In Eq. (44):

$$\frac{d\gamma}{d\tau} = \gamma \frac{d\gamma}{dv} \frac{dv}{dt} = \frac{\gamma^4}{c^2} v \frac{dv}{dt} = \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \quad -(46)$$

therefore the acceleration becomes:

$$\underline{a} = \gamma^4 \frac{d^2 r}{dt^2} \underline{e}_r - \frac{L^2}{m^2 r^3} \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \omega r \underline{e}_\theta \quad -(47)$$

in which the relativistic total angular momentum is defined as:

$$L = \gamma L_0 = m r^2 \frac{d\theta}{d\tau} = \gamma m r^2 \omega \quad -(48)$$

The relativistic force law is therefore the mass m multiplied by the relativistic acceleration:

$$\underline{a} = \left(\gamma^4 \frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \underline{\omega} \times \underline{r} \quad -(49)$$

The term in $\underline{e}_\theta$ can be developed as:

$$\underline{\omega} \times \underline{r} = \omega r \underline{e}_\theta \quad -(50)$$

to arrive at:

$$\underline{a} = \left(\gamma^4 \frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r + \frac{\gamma^4}{c^2} \frac{dr}{dt} \frac{d^2 r}{dt^2} \omega r \underline{e}_\theta \quad -(51)$$

It would be very useful to transform this equation into a format from which it is possible for the relativistic force can be calculated from any observable planar orbit.

This transformation can be deduced following the method developed {1 -10} in previous work for the non relativistic acceleration:

$$\underline{a} = \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m^2 r^3} \right) \underline{e}_r = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) . \quad -(52)$$

This equation is transformed using the chain rules:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} \quad -(53)$$

and

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right) \frac{dr}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} . \quad -(54)$$

Therefore:

$$\frac{dr}{d\theta} = -r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(55)$$

and

$$\frac{dr}{dt} = -r^2 \frac{d}{d\theta}\left(\frac{1}{r}\right) \frac{d\theta}{dt} , \quad -(56)$$

where in the non relativistic theory:

$$\omega = \frac{d\theta}{dt} = \frac{L_0}{m r^2} \quad -(57)$$

so:

$$\frac{dr}{dt} = -\frac{L_0}{m} \frac{d}{d\theta}\left(\frac{1}{r}\right) . \quad -(58)$$

Similarly for any function f:

$$\frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \frac{L_0}{mr^2}\frac{df}{d\theta} \quad -(59)$$

If:

$$f = \frac{dr}{dt} \quad -(60)$$

then:

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d^2r}{dt^2} = \frac{L_0}{mr^2}\frac{d}{d\theta}\left(-\frac{L_0}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right)\right) \quad -(61)$$

so:

$$\frac{d^2r}{dt^2} = -\left(\frac{L_0}{mr}\right)^2\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(62)$$

It follows that the acceleration can be expressed as:

$$\underline{a} = -\left(\frac{L_0}{mr}\right)^2\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \quad -(63)$$

Note carefully that Eq. (63) is valid for ANY planar orbit, and that it includes the centripetal acceleration. The latter is the same for any planar orbit {1 - 10}, and the Coriolis acceleration is zero for any planar orbit.

Eq. (63) is the required equation that gives the non relativistic force for any planar orbit:

$$\underline{F} = m\underline{a} \quad -(64)$$

Q. E. D.

The relativistic counterpart of Eq. (52) is:

$$\underline{a} = \left(\gamma^4\frac{d^2r}{dt^2} - \frac{L^2}{m^2r^3}\right)\underline{e}_r + \frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2r}{dt^2}\,\underline{\omega}\times\underline{r} \quad -(65)$$

where:

$$\underline{\omega} \times \underline{r} = \omega r \, \underline{e}_\theta \quad -(66)$$

and in which the relativistic angular momentum:

$$L = \gamma L_0 = \gamma m r^2 \omega = \gamma m r^2 \frac{d\theta}{dt} \quad -(67)$$

must be used for self consistency. Therefore from Eq. (67):

$$\frac{d\theta}{dt} = \frac{L}{\gamma m r^2} \quad -(68)$$

so:

$$\frac{dr}{dt} = -\frac{L}{m\gamma} \frac{d}{d\theta}\left(\frac{1}{r}\right) \quad -(69)$$

Similarly Eq. (59) becomes:

$$\frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \frac{L}{\gamma m r^2}\frac{df}{d\theta} \quad -(70)$$

so Eq. (62) becomes:

$$\frac{d^2 r}{dt^2} = -\left(\frac{L}{\gamma m r}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(71)$$

$$-(72)$$

Using Eq. (71) in Eq. (65):

$$\underline{a} = -\left(\left(\frac{\gamma L}{m r}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{L^2}{m^2 r^3}\right)\underline{e}_r + \frac{\gamma^4}{c^2}\frac{dr}{dt}\frac{d^2 r}{dt^2}\,\omega r\, \underline{e}_\theta$$

and using Eqs. (68), (69) and (71):

$$\underline{a} = -\left(\frac{L}{m r}\right)^2\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r + \frac{L^4}{m^4 c^2 r^3}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta$$

$$-(73)$$

This is the required equation that gives the relativistic force produced by any planar orbit:

$$\underline{F} = m\underline{a} \quad -(74)$$

Q. E. D.

For the purposes of graphics and animation in Section 3 of this paper it is convenient to express the Lorentz factor in terms of the angle θ. In order to achieve this aim consider the non relativistic velocity of the Lorentz factor in plane polar coordinates

$$\underline{v} = \frac{dr}{dt}\underline{e}_r + \omega r \underline{e}_\theta . \quad -(75)$$

As in preceding papers

$$\underline{v} = \left(\frac{L_0}{m}\right)\left(\frac{1}{r}\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right) \quad -(76)$$

For an elliptical orbit:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos\theta\right) \quad -(77)$$

so:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\sin\theta \quad -(78)$$

and the non relativistic velocity is:

$$\underline{v} = \left(\frac{L_0}{m\alpha}\right)\left(\left(1 + \epsilon\cos\theta\right)\underline{e}_\theta + \epsilon\sin\theta\,\underline{e}_r\right). \quad -(79)$$

Therefore:

$$v^2 = \left(\frac{L_0}{m\alpha}\right)^2\left(1 + \epsilon^2 + 2\epsilon\cos\theta\right) \quad -(80)$$

which is the required expression for v of the Lorentz factor in terms of θ, Q. E. D. Such a procedure can be repeated for any planar orbit in which the dependence of r on θ is known.

The non relativistic velocity v can be expressed in terms of ω and r using Eqs. (75), (76) and the non relativistic total angular momentum:

$$L_0 = mr^2\omega = mr^2 \frac{d\theta}{dt}. \quad -(81)$$

For any planar orbit:

$$\underline{v} = \omega r^2 \left(\frac{1}{r} \underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right) \underline{e}_r \right) \quad -(82)$$

so:

$$v^2 = \omega^2 r^2 \left(1 + r^2 \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 \right) \quad -(83)$$

which is the required expression, Q. E. D. For the elliptical orbit for example:

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(84)$$

so using Eqs. (83) and (84):

$$v^2 = \frac{r^4\omega^2}{\alpha^2}\left(1 + \epsilon^2 + 2\left(\frac{\alpha}{r} - 1\right)\right) \quad -(85)$$

The non relativistic orbital linear velocity of the elliptical orbit is:

$$v^2 = \frac{d\underline{r} \cdot d\underline{r}}{dt^2} \quad -(86)$$

and this is used in the Lorentz factor. In the case of a circular orbit:

$$\epsilon = 0, \quad \alpha = r \quad -(87)$$

and the familiar result is obtained that for a circular orbit in non relativistic theory:

$$v = \omega r. \quad -(88)$$

In summary the relativistic force for any planar orbit is defined by:

$$\underline{F} = -\frac{L^2}{mr^2}\left(\gamma^2\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right)\underline{e}_r + \frac{L^4}{m^3r^3c^2}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta \quad -(89)$$

in which the Lorentz factor is:

$$\gamma = \left(1-\left(\frac{L_0}{mc}\right)^2\left(\frac{1}{r^2}+\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right)\right)^{-1/2} \quad -(90)$$

and in which the relativistic total angular momentum is:

$$L = \gamma L_0 = \gamma m r^2 \frac{d\theta}{dt} = \gamma m r^2 \omega . \quad -(91)$$

Some force laws of various orbits are discussed in Section 3.

For the purposes of animation (Section 3) the analytical expressions for the time evolution is needed for an orbiting object of mass m. In the non relativistic theory of orbits the time evolution is calculated {1 - 11} from the total angular momentum:

$$L_0 = mr^2\frac{d\theta}{dt}. \quad -(92)$$

Therefore:

$$\frac{d\theta}{dt} = \frac{L_0}{mr^2}. \quad -(93)$$

In general the planar orbit is:

$$r = f(\theta). \quad -(94)$$

Therefore:

$$dt = \frac{m}{L_0} f^2(\theta)\, d\theta \quad -(95)$$

which integrates to:

$$t = \int dt = \frac{m}{L_0} \int f^2(\theta)\, d\theta \, . \quad -(96)$$

For the elliptical orbit:

$$f(\theta) = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(97)$$

and for the hyperbolic spiral

$$f(\theta) = -\frac{r_0}{\theta} \, . \quad -(98)$$

Using the chain rule:

$$\frac{d\theta}{dt} = \frac{dr}{dt}\frac{d\theta}{dr} = \frac{L_0}{mr^2} \quad -(99)$$

it is found that:

$$\frac{dr}{dt} = \frac{L_0}{mr^2}\frac{dr}{d\theta} = -\frac{L_0}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right) . \quad -(100)$$

For the hyperbolic spiral (98):

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r_0} \quad -(101)$$

therefore:

$$r = \left(\frac{L_0}{mr_0}\right) t \, . \quad -(102)$$

The relativistic equivalent of Eq. (93) is:

$$\frac{d\theta}{d\tau} = \gamma \frac{L_0}{mr^2} \quad -(103)$$

where

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(104)$$

and

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2\left(1 + r^2\left(\frac{d\theta}{dr}\right)^2\right) \quad -(105)$$

For the hyperbolic spiral (98):

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{L_0}{m r_0}\right)^2 \quad -(106)$$

and

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{r_0^2}{r^4} \quad -(107)$$

so the velocity of the Lorentz factor is given by:

$$v^2 = \left(\frac{L_0}{m r_0}\right)^2\left(1 + \left(\frac{r_0}{r}\right)^2\right) \quad -(108)$$

from which it is found that:

$$v \xrightarrow[r \to \infty]{} \frac{L_0}{m r_0} \quad -(109)$$

as observed experimentally in the well known velocity curve of the whirlpool galaxy.

Therefore the Lorentz factor is:

$$\gamma = \left(1 - \left(\frac{L_0}{m r_0 c}\right)^2\left(1 + \left(\frac{r_0}{r}\right)^2\right)\right)^{-1/2} \quad -(110)$$

Using the chain rules:

$$\frac{d\theta}{d\tau} = \frac{dr}{d\tau}\frac{d\theta}{dr} = \frac{\gamma L_0}{m r^2} \quad -(111)$$

and

$$\frac{dr}{d\tau} = \frac{\gamma L_0}{mr^2} \cdot \frac{dr}{d\theta} \quad -(112)$$

it is found that:

$$r = \frac{\gamma L_0}{mr_0} \tau \quad -(113)$$

In the frame of the proper time τ, r is proportional to $\gamma\tau$. The proper time is the time measured in the frame in which the particle is at rest. The observer or laboratory time t is the time in the frame with respect to which the particle of mass m is moving. By definition of the Lorentz transform:

$$\gamma = \frac{dt}{d\tau} \quad -(114)$$

so

$$\tau = \frac{t}{\gamma} \quad -(115)$$

and

$$\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} t \quad -(116)$$

Therefore from Eqs. (103), (110) and (113):

$$r = \left(1 - \left(\frac{L_0}{mr_0 c}\right)^2 \left(1 + \left(\frac{r_0}{r}\right)^2\right)\right)^{1/2} \frac{L_0}{mr_0} \tau \quad -(117)$$

and in the relativistic theory of the hyperbolic spiral orbit there is no longer a simple dependence of r on t. Solving for r (see note 238(9)):

$$r = \frac{\tau}{\sqrt{2}} \left(A\tau + \left(A^2\tau^2 - 4B\right)^{1/2}\right)^{1/2} \quad -(118)$$

where:

$$A = \left(1 - \left(\frac{L_0}{m r_0 c}\right)^2\right)\left(\frac{L_0}{m r_0}\right)^2 \quad -(119)$$

and

$$B = \left(\frac{L_0}{m r_0 c}\right)^2\left(\frac{L_0}{m r_0}\right)^2 r_0^2 \quad -(120)$$

This equation reduces to Eq. (102) in the non relativistic limit v << c.

In order to animate the motion of a mass m along a hyperbolic spiral trajectory it is convenient to transform coordinates. Starting from:

$$\underline{r}(t) = r(t)\,\underline{e}_r \quad -(121)$$

express the unit vector as:

$$\underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \quad -(122)$$

so:

$$\underline{r}(t) = r(t)\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}\right) \quad -(123)$$

For the hyperbolic spiral:

$$r = -r_0/\theta, \quad -(124)$$

$$\theta = -r_0/r, \quad -(125)$$

so:

$$\begin{aligned}\underline{r}(t) &= r(t)\left(\cos\left(-\frac{r_0}{r}\right)\underline{i} + \sin\left(-\frac{r_0}{r}\right)\underline{j}\right)\\ &= r(t)\left(\cos\left(\frac{r_0}{r}\right)\underline{i} - \sin\left(\frac{r_0}{r}\right)\underline{j}\right)\end{aligned} \quad -(126)$$

In the non relativistic limit:

$$r = \left(\frac{L_0}{m r_0}\right) t \quad -(127)$$

so:

$$\underline{r}(t) = \left(\frac{L_0}{m r_0}\right) t \left(\cos\left(\frac{m r_0^2}{L_0 t}\right) \underline{i} - \sin\left(\frac{m r_0^2}{L_0 t}\right) \underline{j} \right) \quad -(128)$$

The animation proceeds as a series of (X, Y) plots as a function of time t, where:

$$X = \left(\frac{L_0}{m r_0}\right) t \cos\left(\frac{m r_0^2}{L_0 t}\right), \quad -(129)$$

$$Y = \left(\frac{L_0}{m r_0}\right) t \sin\left(\frac{m r_0^2}{L_0 t}\right). \quad -(130)$$

The circular functions have the well known property:

$$-1 \leq \cos\theta \leq 1, \quad -(131)$$

$$-1 \leq \sin\theta \leq 1, \quad -(132)$$

so as

$$t \rightarrow 0 \quad -(133)$$

in Eqs. (10) and (11):

$$X \xrightarrow[t \to 0]{} 0, \quad Y \xrightarrow[t \to 0]{} 0 \quad -(134)$$

and as

$$t \rightarrow \infty \quad -(135)$$

then:

$$X \rightarrow \infty \quad -(136)$$

$$Y \rightarrow \text{constant}, \quad -(137)$$

and the particle moves outward along the spiral from the centre of the galaxy.

To animate the relativistic motion of a mass m along the hyperbolic spiral use the relativistic Eq. (118) in Eq. (126).

The animation of the motion of a mass m along the elliptical orbit is a non trivial problem analytically but can be coded with the following method. In this case the non relativistic theory gives:

$$t = \int dt = \frac{m\alpha^2}{L_0} \int \frac{d\theta}{(1+\epsilon\cos\theta)^2} \quad -(138)$$

$$= \frac{m\alpha^2}{L_0(1-\epsilon^2)} \left[\frac{2}{(1-\epsilon^2)^{1/2}} \tan^{-1}\left(\frac{(1-\epsilon)\tan(\theta/2)}{(1-\epsilon^2)^{1/2}} \right) - \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta} \right] \quad -(139)$$

and as is well known {11} this equation cannot be inverted analytically to find θ as a function of time t. The trajectory as a function of θ is:

$$\underline{r}(\theta) = r(\theta)(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) \quad -(140)$$

where:

$$r(\theta) = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(141)$$

Therefore:

$$X = \frac{\alpha\cos\theta}{1+\epsilon\cos\theta}, \quad Y = \frac{\alpha\sin\theta}{1+\epsilon\cos\theta} \quad -(142)$$

and plot the points (X, Y) as a function of θ (see note 238(12)). The points (X, Y) can be plotted as a function of t as required for an animation by using Eq. (139) to transform θ into t.

For example:

$$t_1 = \frac{m\alpha^2}{L_0}\int_0^{\theta_1}\frac{d\theta}{(1+\epsilon\cos\theta)^2} \quad -(143)$$

$$= \frac{m\alpha^2}{L_0(1-\epsilon^2)}\left[\frac{2}{(1-\epsilon^2)^{1/2}}\tan^{-1}\left(\frac{(1-\epsilon)\tan(\theta_1/2)}{(1-\epsilon^2)^{1/2}}\right) - \frac{\epsilon\sin\theta_1}{1+\epsilon\cos\theta_1}\right]$$

This method gives the correct speed and obeys the three Kepler laws automatically.

For the precessing ellipse or conical section the non relativistic theory gives:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(144)$$

and:

$$t = \frac{m\alpha^2}{L_0}\int\frac{d\theta}{(1+\epsilon\cos(x\theta))^2} \quad -(145)$$

so:

$$X = \frac{\alpha\cos\theta}{1+\epsilon\cos(x\theta)}, \quad Y = \frac{\alpha\sin\theta}{1+\epsilon\cos(x\theta)} \quad -(146)$$

and plot (X, Y) as a function of time t. It is known from previous work {1 - 10} that fractal conical sections emerge when x is increased, and the animation of this type of orbit will give very interesting results in mathematics and astronomy.

The relativistic theory of the conical section orbits gives:

$$\frac{d\theta}{d\tau} = \gamma\frac{L_0}{mr^2} \quad -(147)$$

in which the Lorentz factor is:

$$\gamma = \frac{dt}{d\tau} = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(148)$$

with velocity defined by:

$$\underline{v} = \frac{L_0}{m}\left(\frac{1}{r}\,\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right) \qquad -(149)$$

From Eq. (147):

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \frac{m r^2}{L_0}\, d\theta \qquad -(150)$$

and integrating gives:

$$\tau = \frac{m}{L_0}\int \left(1 - \frac{v^2}{c^2}\right)^{1/2} r^2\, d\theta \qquad -(151)$$

For the conical sections Eq. (80) gives the velocity of the Lorentz factor as a function of θ

as:

$$v^2 = \left(\frac{L_0}{m d}\right)^2 \left(1 + \epsilon^2 + 2\epsilon\cos\theta\right) \qquad -(152)$$

so:

$$\tau = \frac{m d^2}{L_0}\int \left(1 - \left(\frac{L_0}{m d c}\right)^2 \left(1 + \epsilon^2 + 2\epsilon\cos\theta\right)\right)^{1/2} \frac{d\theta}{(1 + \epsilon\cos\theta)^2} \qquad -(153)$$

which must be integrated numerically. The animation proceeds by using Eq. (153) and calculating τ for each θ . For example:

$$\tau_1 = \frac{m}{L_0}\int_0^{\theta_1} \left(1 - \left(\frac{L_0}{m d c}\right)^2 \left(1 + \epsilon^2 + 2\epsilon\cos\theta\right)\right)^{1/2} \frac{d\theta}{(1 + \epsilon\cos\theta)^2} \qquad -(154)$$

Repeat for t , t , ..., t and animate (X, Y) as a function of t.

Finally the relativistic theory of the precessing conical sections proceeds with

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(155)$$

and:

$$\tau = \frac{m\alpha^2}{L_0}\int\left(1-\frac{v^2}{c^2}\right)^{1/2}\frac{d\theta}{(1+\epsilon\cos(x\theta))^2} \quad -(156)$$

The velocity of the Lorentz factor is:

$$\underline{v} = \frac{L_0}{m}\left(\frac{(1+\epsilon\cos(x\theta))}{\alpha}\,\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right) \quad -(157)$$

where:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{x\epsilon}{\alpha}\sin(x\theta) \quad -(158)$$

so

$$v^2 = \left(\frac{L_0}{m\alpha}\right)^2\left((1+\epsilon\cos(x\theta))^2 + x^2\epsilon^2\sin^2(x\theta)\right) \quad -(159)$$

Therefore τ is computed from Eqs. (156) and (159) for a given θ , for example:

$$\tau_1 = \frac{m\alpha^2}{L_0}\int_0^{\theta_1}\left(1-\frac{v^2}{c^2}\right)^{1/2}\frac{d\theta}{(1+\epsilon\cos(x\theta))^2} \quad -(160)$$

with:

$$v^2 = \left(\frac{L_0}{m\alpha}\right)^2\left((1+\epsilon\cos(x\theta))^2 + x^2\epsilon^2\sin^2(x\theta)\right). \quad -(161)$$

3. DISCUSSION OF GRAPHICAL RESULTS AND ANIMATIONS

Section by Horst Eckardt and Robert Cheshire.

ACKNOWLEDGMENTS

The British Government is thanked for a Civil List Pension and the AIAS and others for discussions. Dave Burleigh is thanked for posting and Alex Hill and Robert Cheshire for translations and broadcasting. AIAS is established in the Trust Law of England and Wales under the Newlands Family Trust (Est. 2012). It is closely associated with UPITEC, which is a not for profit organization in Boise, Idaho, U. S. A.

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Applications of ECE theory: the relativistic kinematics of orbits

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3 Discussion of graphical results and animations

In this section we gather the formulas for the calculation of orbits and then discuss some results for the hyperbolic spiral and the precessing ellipse.

3.1 The formulae

3.1.1 The non-relativistic formulae

An orbit is given by the function

$$r = r(\theta), \tag{162}$$

the orbital velocity is

$$v = \frac{L_0}{m}\sqrt{\frac{1}{r^2} + \left(\frac{d}{d\theta}\,\frac{1}{r}\right)^2}, \tag{163}$$

the force components are

$$F_r = -\left(\frac{L^2}{m\,r^2}\left(\frac{d^2}{d\,\theta^2}\,\frac{1}{r}\right) + \frac{L^2}{m\,r^3}\right), \tag{164}$$

$$F_\theta = \frac{L^4}{c^2\,m^3\,r^3}\left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)\left(\frac{d^2}{d\,\theta^2}\,\frac{1}{r}\right) \tag{165}$$

with $L = L_0$ being the non-relativistic angular momentum, a constant of motion. The time dependencies are calculated as follows:

$$t(\theta) = \frac{m}{L_0}\int r(\theta)^2\,d\theta, \tag{166}$$

$$\theta(t) = \text{inverse function of } t(\theta,) \tag{167}$$

$$r(t) = r(\theta(t)), \text{ follows directly from orbital function } r(\theta). \tag{168}$$

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3.1.2 The relativistic formulae

The relativistic γ factor is

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1 - \frac{{L_0}^2}{c^2\,m^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)^2\right)\right)^{-1/2}. \tag{169}$$

The force components are

$$\begin{aligned}
F_r =& -\left(\gamma^2\,\frac{L^2}{m\,r^2}\left(\frac{d^2}{d\,\theta^2}\,\frac{1}{r}\right) + \frac{L^2}{m\,r^3}\right) && (170)\\
=& -\frac{{L_0}^2}{m\,r^2}\left(\frac{d^2}{d\,\theta^2}\,\frac{1}{r}\right)\left(1 - \frac{{L_0}^2}{c^2\,m^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)^2\right)\right)^{-2}\\
& -\frac{{L_0}^2}{m\,r^3}\left(1 - \frac{{L_0}^2}{c^2\,m^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)^2\right)\right)^{-1},\\
F_\theta =& \frac{L^4}{c^2\,m^3\,r^3}\left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)\left(\frac{d^2}{d\,\theta^2}\,\frac{1}{r}\right) && (171)\\
=& \frac{{L_0}^4}{c^2\,m^3\,r^3}\left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)\left(\frac{d^2}{d\,\theta^2}\,\frac{1}{r}\right)\left(1 - \frac{{L_0}^2}{c^2\,m^2}\left(\frac{1}{r^2} + \left(\frac{d}{d\,\theta}\,\frac{1}{r}\right)^2\right)\right)^{-2}.
\end{aligned}$$

The γ factor contains $r(\theta)$ which in turn is a function of time. Therefore the γ factor is part of the following integral:

$$t(\theta) = \frac{m}{L_0}\int \gamma\, r(\theta)^2\, d\theta, \tag{172}$$

$$\theta(t) = \text{inverse function of } t(\theta), \tag{173}$$

$$r(t) = r(\theta(t)),\ \text{follows directly from orbital function } r(\theta). \tag{174}$$

3.2 Results for the hyperbolic spiral

The hyperbolic orbit is

$$r = -\frac{r_0}{\theta}. \tag{175}$$

θ runs from $-\infty$ to 0 so the radius values are positive. A more natural definition would be not to use the minus sign but then Eq.(172) gives a negative time. This could be remedied by using the absolute value of the integral.

3.2.1 Non-relativistic results

From the formulas above we obtain with Eq.(175):

$$v = \frac{L_0}{m\,r_0}\sqrt{\theta^2+1}. \tag{176}$$

For $\theta \to 0$ this approaches the constant asymptotic velocity curve of galaxies:

$$v \to \frac{L_0}{m\,r_0}. \tag{177}$$

If the angular momentum of a star in the constant velocity region can be method and its mass is known, an estimation for the spiral parameter r_0 is possible.

The other quantities are linear or hyperbolic in t:

$$F_r = \frac{\theta^3 \, {L_0}^2}{m \, {r_0}^3}, \tag{178}$$

$$F_\theta = 0, \tag{179}$$

$$t = -\frac{m \, {r_0}^2}{\theta \, L_0}, \tag{180}$$

$$\theta = -\frac{m \, {r_0}^2}{t \, L_0}, \tag{181}$$

$$r = \frac{t \, L_0}{m \, r_0}. \tag{182}$$

3.2.2 Relativistic results

For the hyperbolic spiral the relativistic γ factor is

$$\gamma = \frac{1}{\sqrt{1 - \frac{{L_0}^2}{c^2 \, m^2 r_0^2} \left(\theta^2 + 1\right)}} \tag{183}$$

and the forces are

$$F_r = -\frac{\theta^3 \, {L_0}^2}{m \, {r_0}^3} \left(1 - \frac{{L_0}^2}{c^2 \, m^2 r_0^2} \left(\theta^2 + 1\right)\right)^{-1}, \tag{184}$$

$$F_\theta = 0. \tag{185}$$

Obviously there is no angular deviation due to relativistic effects. The functions t, θ and r are quite complicated due to the γ factor:

$$\begin{aligned} t = & -\frac{m \, {r_0}^2 \sqrt{-\frac{\theta^2 \, {L_0}^2}{c^2 \, m^2 \, {r_0}^2} - \frac{{L_0}^2}{c^2 \, m^2 \, {r_0}^2} + 1}}{\theta \, L_0 \left(1 - \frac{{L_0}^2}{c^2 \, m^2 \, r0^2}\right)} \\ = & \frac{c \, m^2 \, {r_0}^3 \sqrt{c^2 \, m^2 \, {r_0}^2 - \left(\theta^2 + 1\right) \, {L_0}^2}}{\theta \, L_0 \, \left(L_0 - c \, m \, r_0\right) \, \left(L_0 + c \, m \, r_0\right)}, \end{aligned} \tag{186}$$

$$\theta = -\frac{c \, m^2 \, {r_0}^3}{L_0} \sqrt{\frac{c^2 \, m^2 \, {r_0}^2 - {L_0}^2}{t^2 \, {L_0}^4 - 2 \, c^2 \, m^2 \, {r_0}^2 \, t^2 \, {L_0}^2 + c^4 \, m^4 \, {r_0}^4 \, t^2 + c^2 \, m^4 \, {r_0}^6}}, \tag{187}$$

$$r = -\frac{r_0}{\theta} = \frac{L_0}{c \, m^2 \, {r_0}^2} \sqrt{\frac{t^2 \, {L_0}^4 - 2 \, c^2 \, m^2 \, {r_0}^2 \, t^2 \, {L_0}^2 + c^4 \, m^4 \, {r_0}^4 \, t^2 + c^2 \, m^4 \, {r_0}^6}{c^2 \, m^2 \, {r_0}^2 - {L_0}^2}}. \tag{188}$$

These equations can be rewritten to the form

$$\theta = -\frac{m \, r_0^2}{t \, L_0} \cdot C(t), \tag{189}$$

$$r = \frac{t \, L_0}{m \, r_0} / C(t) \tag{190}$$

with time-dependent functions

$$C(t) = \sqrt{\frac{1-a}{1-2a+a^2+b(t)}}, \quad a = \left(\frac{L_0}{c\,r_0\,m}\right)^2, \quad b(t) = \left(\frac{r_0}{c\,t}\right)^2. \tag{191}$$

In this way it can be seen that $C(t)$ is a relativistic correction factor to the non-relativistic Eqs. (181, 182). In the limit $t \to 0$ the functions θ and r do not approach infinity or zero as in the non-relativistic case. The limits are

$$\theta(t \to 0) = -\frac{\sqrt{c^2\,m^2\,{r_0}^2 - {L_0}^2}}{L_0}, \tag{192}$$

$$r(t \to 0) = \frac{r_0\,L_0}{\sqrt{c^2\,m^2\,{r_0}^2 - {L_0}^2}}. \tag{193}$$

For the square root argument to be positive we have the condition

$$L_0 < c\,m\,r_0. \tag{194}$$

This is the same condition as obtained from requesting $v < c$ for the velocity curve, Eq.(177). It has however to be observed that the velocity near to the center is highest, therefore the true condition $v < c$ has a much lower value of L_0. In the ultrarelativistic limit the spiral starts distinctly from the center as can be seen in the following graphs and animation section.

The elliptic orbit is graphed in Fig. 1. The ratio v/c is shown in Fig. 2 in a polar plot. This ratio is not constant as in the simple form of Einstein special relativity but depends on the angle-dependent orbital velocity. In the calculations we set $m = c = r_0 = 1$ so the orbital velocity as well as the ratio v/c is defined by the angular momentum L_0. For all calculations we chose

$$L_1 = 0.01,\ L_2 = 0.03,\ L_3 = 0.06,\ L_4 = 0.069 \tag{195}$$

so that v/c remains below 1. From Fig. 3 it can be seen that the radial force increases significantly in the relativistic case. In Fig. 4 the region near to the centre is shown in an enlarged image. Obviously the directional characteristic is changed by relativistic effects. The time dependence effects are depicted by Figs. 5 and 6. The angle starts at significantly lower absolute values in the relativistic case as expected from the formulas. This corresponds to a larger start radius (Fig. 6). Both the non-relativistic and relativistic orbit are the same by definition.

3.2.3 Animations

Both orbits were animated acoording to the formulas given. The animation is done for the time-dependent orbital vector

$$\mathbf{r}(t) = [X(t), Y(t)] \tag{196}$$

which in cartesian coordinates is given by

$$X(t) = r(t)\cos\left(\frac{r_0}{r(t)}\right), \tag{197}$$

$$Y(t) = r(t)\sin\left(\frac{r_0}{r(t)}\right), \tag{198}$$

where $r(t)$ is Eq.(182) or (188) in the non-relativistic or relativistic case, respectively. The screenshots Figs. 7 and 8 give an impression of the animation program. A large number of parameters can be set like number of spiral arms and variations in angle and time of star production in order to give a more realistic impression of a spiral galaxy. Relativistic motion can be combined with non-relativistic motion. The start position of relativistic motion can be seen nicely. For studying the constant velocity curve, parameters can be set similar to Fig. 8, for example end angle 0.05 and animation speed 20.

The animation program is written as an application for Microsoft Windows can be downloaded from the AIAS website []. After the first run a configuration file Galaxie.CNF is written which saves window size and position and is read again at the beginning of the next run.

3.3 Results for the precessing ellipse

For the precessing ellipse the orbit is

$$r = \frac{\alpha}{\epsilon\cos(\theta\,x)+1}. \tag{199}$$

3.3.1 Non-relativistic results

The non-relativistic results for this orbit are

$$v = \frac{L_0}{\alpha\,m}\sqrt{\epsilon^2\,x^2\sin(\theta\,x)^2+\epsilon^2\cos(\theta\,x)^2+2\,\epsilon\cos(\theta\,x)+1}, \tag{200}$$

$$F_r = \frac{\epsilon\,x^2\cos(\theta\,x)\,\left(\epsilon\cos(\theta\,x)+1\right)^2 {L_0}^2}{\alpha^3\,m} - \frac{\left(\epsilon\cos(\theta\,x)+1\right)^3 {L_0}^2}{\alpha^3\,m}, \tag{201}$$

$$F_\theta = \frac{\epsilon^2\,x^3\cos(\theta\,x)\,\left(\epsilon\cos(\theta\,x)+1\right)^3\sin(\theta\,x)\,{L_0}^4}{\alpha^5\,c^2\,m^3}, \tag{202}$$

$$t = \frac{2\alpha^2 m}{x\,L_0}\left(\frac{\operatorname{atan}\left(\frac{(2\,\epsilon-2)\sin(\theta\,x)}{2\sqrt{1-\epsilon^2}\,(\cos(\theta\,x)+1)}\right)}{\sqrt{1-\epsilon^2}\,(\epsilon^2-1)} - \frac{\epsilon\sin(\theta\,x)}{(\cos(\theta\,x)+1)\left(\frac{(\epsilon^3-\epsilon^2-\epsilon+1)\sin(\theta\,x)^2}{(\cos(\theta\,x)+1)^2}-\epsilon^3-\epsilon^2+\epsilon+1\right)}\right). \tag{203}$$

The function $t(\theta)$ cannot be inverted, therefore an analytic dependence $\theta(t)$ and $r(t)$ cannot be given. These have to be calculated numerically.

3.3.2 Relativistic results

In the relativistic case computer algebra obtains

$$\gamma = \frac{1}{\sqrt{1 - \frac{L_0{}^2}{c^2\, m^2 \alpha^2} \left(\epsilon^2\, x^2 \sin\left(\theta\, x\right)^2 + \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^2\right)}}, \tag{204}$$

$$\begin{aligned} F_r =& \frac{L_0{}^2\, \epsilon\, x^2 \cos\left(\theta\, x\right)\, \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^2}{\alpha^3\, m \left(1 - \frac{L_0{}^2}{c^2\, m^2 \alpha^2} \left(\epsilon^2\, x^2 \sin\left(\theta\, x\right)^2 + \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^2\right)\right)^2} \\ & - \frac{L_0{}^2 \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^3}{\alpha^3\, m \left(1 - \frac{L_0{}^2}{c^2\, m^2 \alpha^2} \left(\epsilon^2\, x^2 \sin\left(\theta\, x\right)^2 + \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^2\right)\right)}, \end{aligned} \tag{205}$$

$$F_\theta = \frac{L_0{}^4\, \epsilon^2\, x^3 \cos\left(\theta\, x\right)\, \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^3 \sin\left(\theta\, x\right)}{\alpha^5\, c^2\, m^3 \left(1 - \frac{L_0{}^2}{c^2\, m^2 \alpha^2} \left(\epsilon^2\, x^2 \sin\left(\theta\, x\right)^2 + \left(\epsilon \cos\left(\theta\, x\right) + 1\right)^2\right)\right)^2}. \tag{206}$$

Obviously there is an angular component of the force, in contrast to the hyperbolic spiral. The first plot of the precessing ellipse (Fig. 9) is the orbit, for clarity only two roundations are shown for $x = 1.1$. All other parameters were set to unity except for the angular momentum as described above. Reasonable values of $v < c$ are obtained for

$$L_1 = 0.1,\ L_2 = 0.3,\ L_3 = 0.6,\ L_4 = 0.69. \tag{207}$$

The ratio v/c is shown in Fig. 10 for these values. The radial force component ist graphed in a cartesion plot in Fig. 11. For the highest L value, representing the ultrarelativistic case, F_r takes positive values, i.e. we have antigravity effects near to the perihelion, which is a quite astonishing result. The angular component of the Minkowski force is much smaller as can be seen from Fig. 12. It is symmetric to the point $\theta = \pi$ so that its line integral vanishes, i.e. F_θ evokes some accelerations which cancel out in average over the orbit. In Figs. 13 and 14 the absolute values of both force components are graphed in a polar plot, showing the form of juggling clubs because of the zero crossings of the forces. Finally the non-relativistic time function $t(\theta)$ is presented in Fig. 15. The jump is an artefact of the arctan function which produces values in the interval $[-\pi/2, \pi/2]$ by definition.

We conclude that relativistic effects lead to quite complicated force laws. All orbital functions can only be calculated analytically for the hyperbolic spiral. Tests with other types of spirals showed that they cannot be handled fully anaytically as is the case for the precessing ellipse.

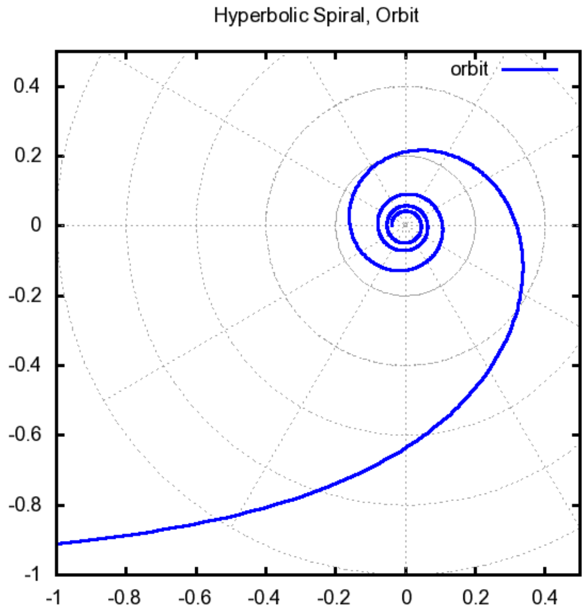

Figure 1: Hyperbolic spiral orbit wiht $r_0 = 1$.

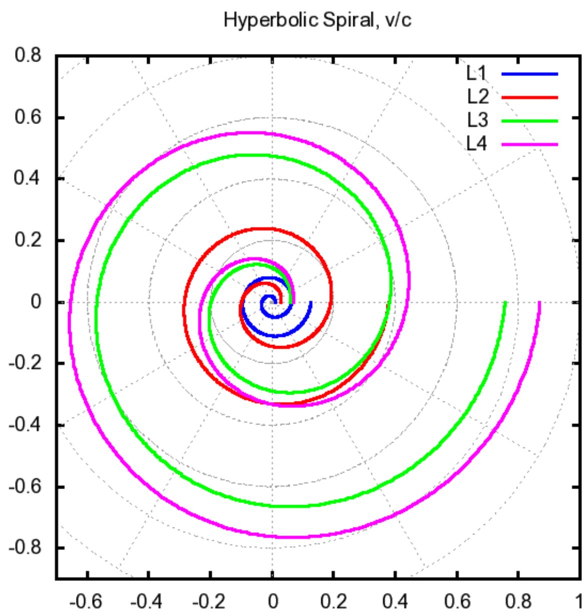

Figure 2: Ratio v/c of the hyperbolic spiral for several values of angular momentum.

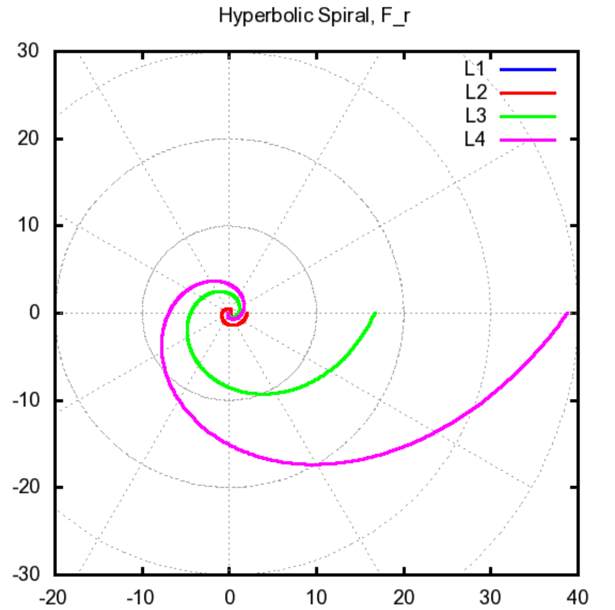

Figure 3: Radial force component of the hyperbolic spiral.

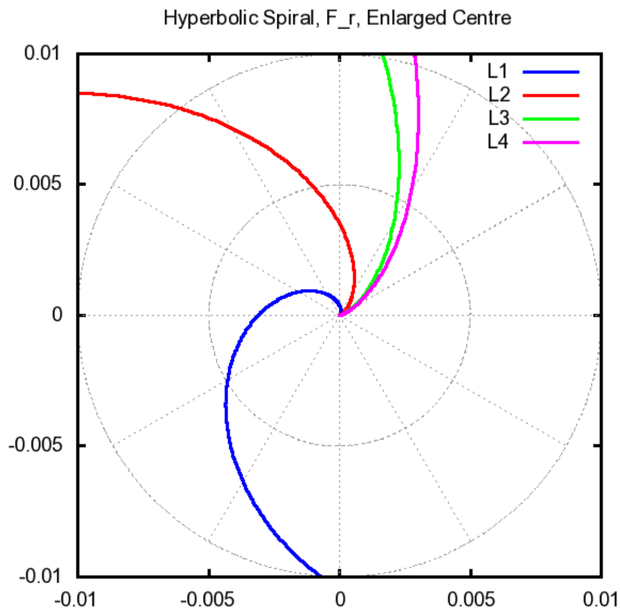

Figure 4: Radial force component of the hyperbolic spiral in the region near to the centre.

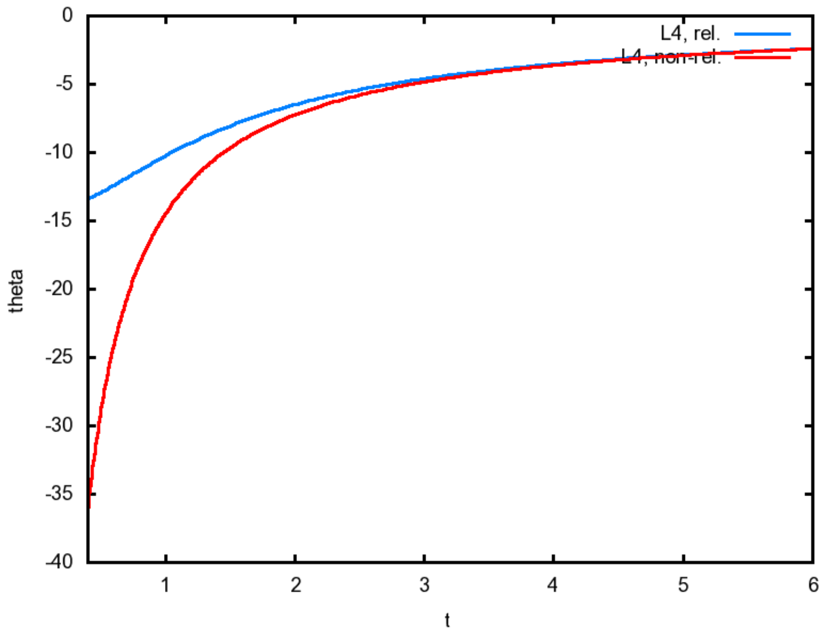

Figure 5: Time dependence $\theta(t)$ for the hyperbolic spiral in relativistic and non-relativistic case.

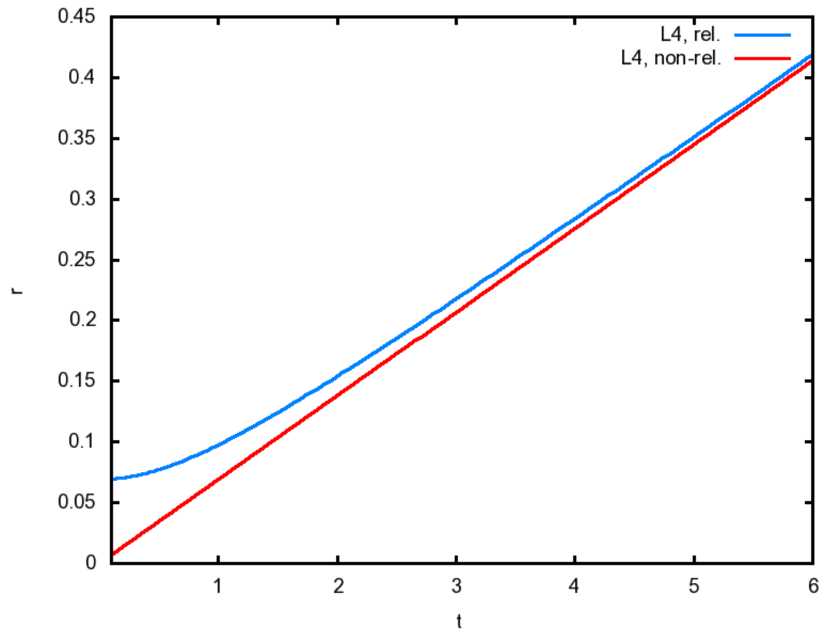

Figure 6: Radius dependence $r(t)$ for the hyperbolic spiral in relativistic and non-relativistic case.

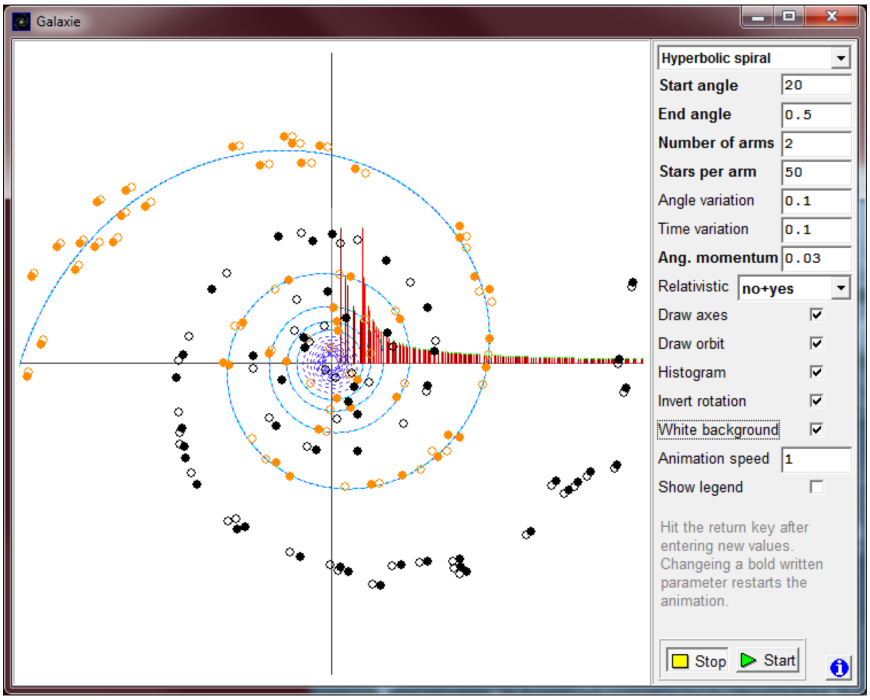

Figure 7: Screenshot of the animation program for hyperbolic spirals.

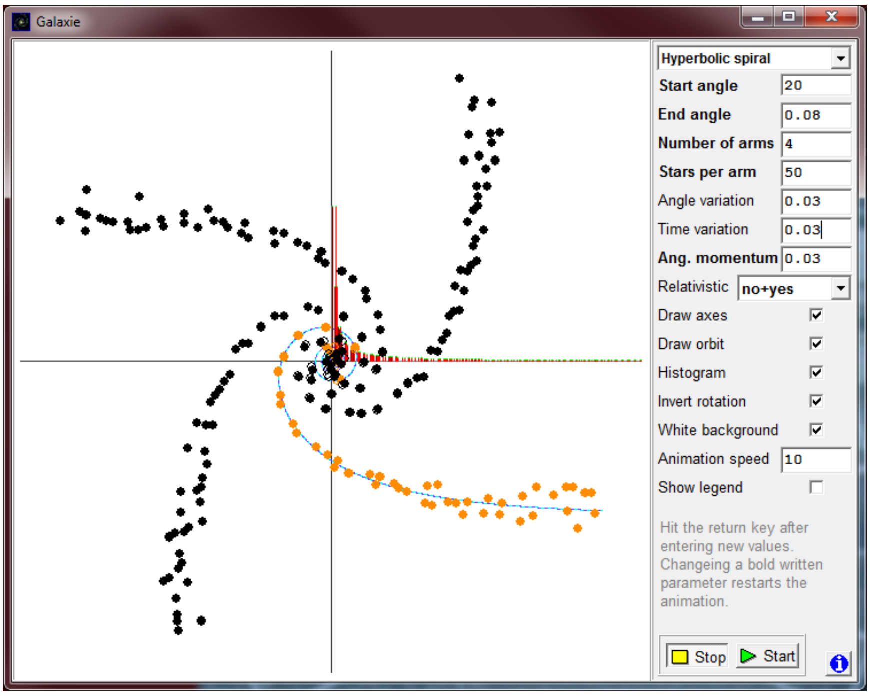

Figure 8: Another screenshot of the animation program for hyperbolic spirals.

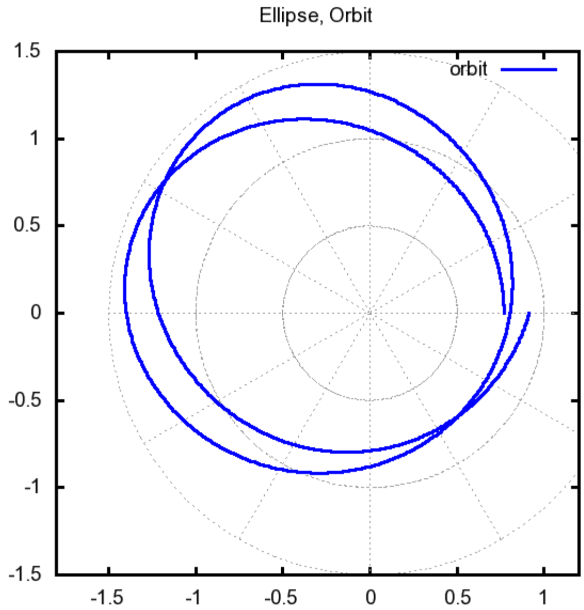

Figure 9: Orbit of a precessing ellipse with $x = 1.1$.

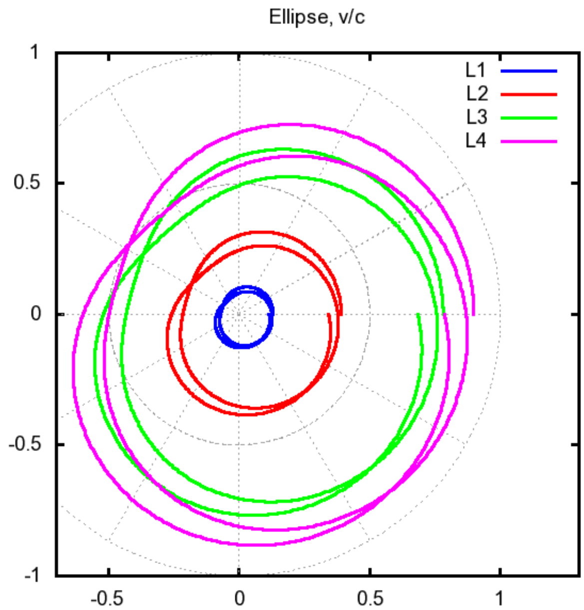

Figure 10: Ratio v/c for the precessing ellipse for several values of angular momentum.

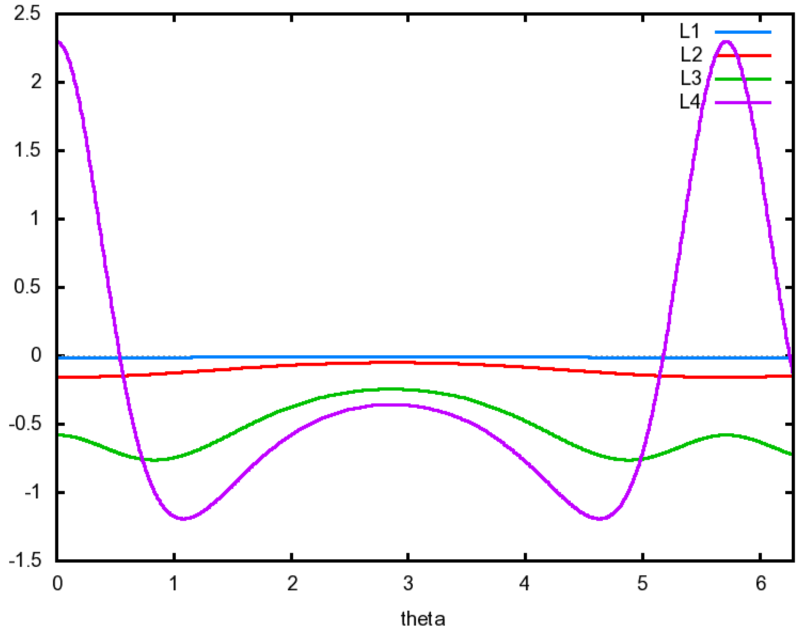

Figure 11: Radial force component of the precessing ellipse, cartesian plot.

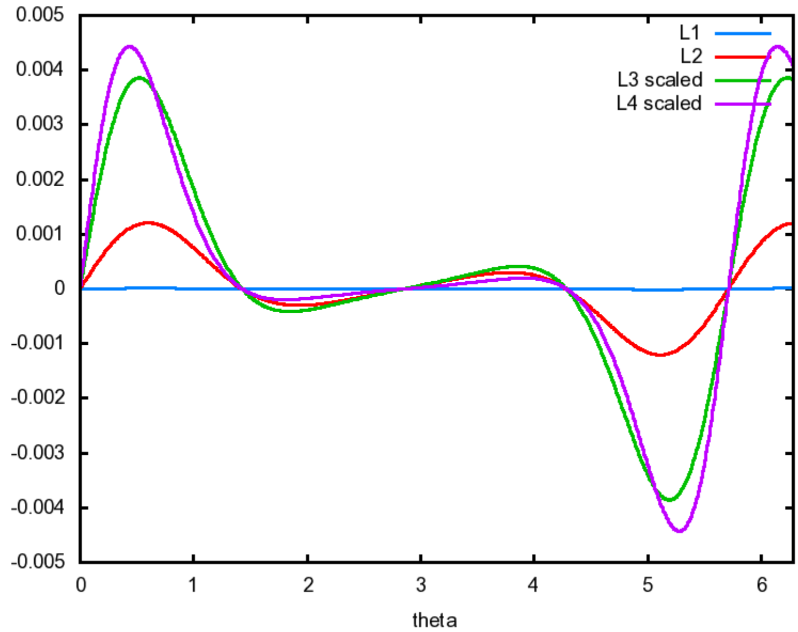

Figure 12: Angular force component of the precessing ellipse, cartesian plot.

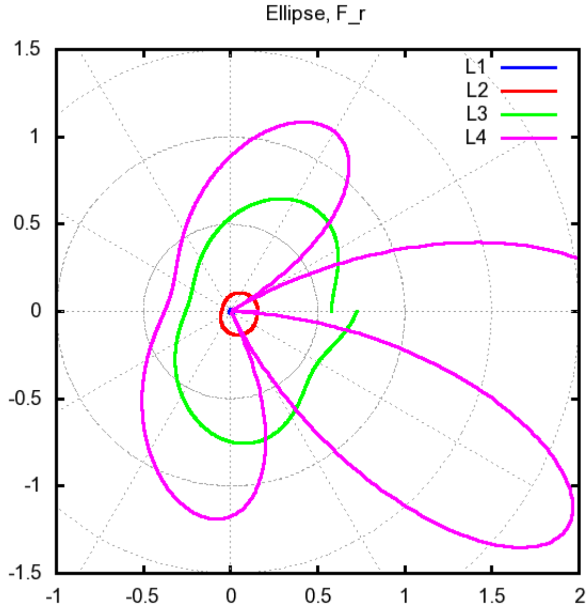

Figure 13: Radial force component of the precessing ellipse, polar plot.

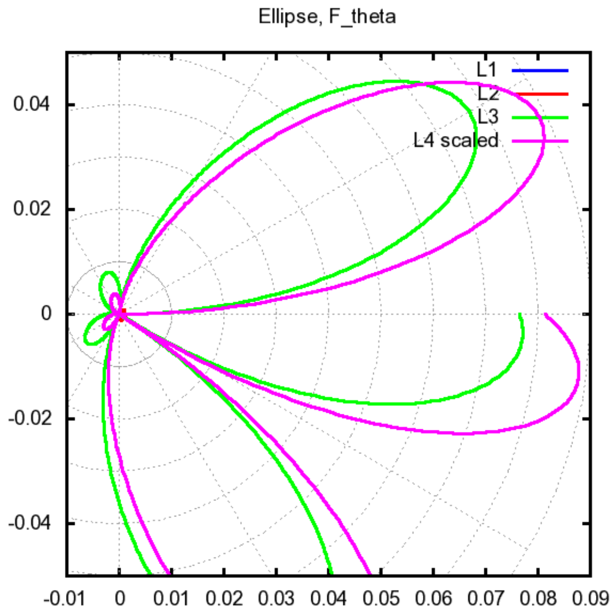

Figure 14: Angular force component of the precessing ellipse, polar plot.

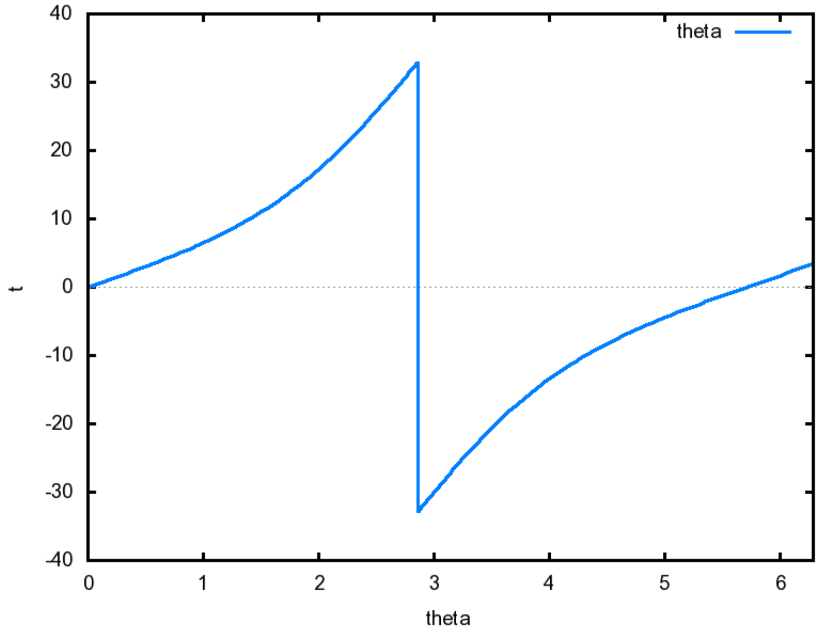

Figure 15: Non-relativistic angular time-dependence $t(\theta)$ of the precessing ellipse.