ORBITALS OF THE BOHR AND SOMMERFELD ATOMS WITH QUANTIZED x THEORY

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ABSTRACT

It is shown that the energy levels of the Bohr atom, and the Bohr radii of the atom, are given by a static ellipse of x theory in the limit of vanishing eccentricity. The Sommerfeld atom is the relativistic correction to the Bohr atom and its orbits are shown to be precessing ellipses of x theory. In the transition from the Bohr theory to the Sommerfeld theory the velocity of the Lorentz transform remains constant, and is the Bohr velocity. The energy levels and other features of the Sommerfeld atom can be calculated straightforwardly from x theory, which is ECE theory in which the angular velocity is the spin connection. A novel quantization scheme is introduced in which the precession factor x is equated with the principal quantum number n.

Keywords: ECE theory, x theory of the Bohr and Sommerfeld atoms.

4FT 266

INTRODUCTION

In recent papers of this series {1 - 10} the x theory of planar orbits has been developed from ECE theory with the result that several well known phenomena of planetary precession have been explained to state of the art accuracy. The x theory of planetary precession is based directly on the experimentally observed precession of an ellipse, and the origin of the precession shown to be the rotation of the Minkowski metric in the Thomas precession (see notes 1 and 2 accompanying UFT266 on www.aias.us). The same theory has been used self consistently to explain electromagnetic deflection due to gravitation and gravitational time delay. It was also used to deduce relativisic photon velocity and photon mass. The gravitational red shift was explained using the Minkowski metric of x theory, in which the velocity of the Lorentz factor is the orbital velocity. In general this is not a constant. The same x theory was used in UFT264 to refute the Einstein theory of gravitation by showing that it produces an infinity. The correct force law of precessing orbitals is given by x theory using the Binet equation. It gives the Leibniz force law of 1689 multiplied by the square of x. The Einstein theory gives a different force law which can never give a precessing ellipse.

In Section 2 The Bohr and Sommerfeld theories of the atom are developed in terms of static and precessing ellipses of x theory. The Bohr theory of 1913 is shown to be the limit of an x theory ellipse whose half right latitude gives the Bohr radii immediately upon quantization. This result is true for any ellipticity. The well known energy levels of the Bohr atom are given immediately in the limit of vanishing ellipticity, when the ellipse of x theory becomes a circle, the well known circular orbits of the Bohr atom. The Sommerfeld theory of the atom developed in 1915 was the first relativistic quantum theory in which a hamiltonian was constructed from the sum of the relativistic kinetic energy and the Coulombic energy of attraction. It is shown in Section two that the Sommerfeld atom is consistent with a precessing ellipse, whose characteristics are slightly different from circular because the relativistic corrections are small. The linear orbital velocity in the Sommerfeld atom is the same as that on the Bohr atom, and is the Bohr velocity. This result follows from the Minkowski metric used in the Sommerfeld theory. The latter is developed both from the hamiltonian and relativistic force law. Finally in Section 2 a novel quantization scheme is introduced in which the precession factor x of the ellipse is equated to the principal quantum number n. This Eckardt quantization results in the appearance of wave structure superimposed on the ellipse, and is discussed further in Section 3. In Section 3 the characteristics of the Sommerfeld atom are analyzed numerically to give novel results from the Sommerfeld theory of the atom.

2. QUANTIZATION OF x THEORY

The Bohr theory of atomic hydrogen is described by the force law:

$$m\frac{d^2 r}{dt^2} = -\frac{e^2}{4\pi\epsilon_0 r^2} + \frac{L^2}{mr^3} = 0 \quad -(1)$$

where m is the mass of the electron orbiting the proton, and where -e is the charge on the electron. Here L is the conserved total angular momentum, ϵ_0 is the vacuum permittivity in S. I. units and where r is the distance between electron and proton. The original Bohr atom of 1913 was a theory which assumed circular orbits:

$$\frac{dr}{dt} = 0 \quad -(2)$$

so from Eq. (1):

$$\frac{L^2}{mr^3} = \frac{e^2}{4\pi\epsilon_0 r^2}. \quad -(3)$$

Bohr assumed Eq. (3) ad hoc, or heuristically, but it is in fact the result of assuming circular orbits in Eq. (1). The latter is precisely analogous to the 1689 Leibniz equation of orbits, still used today:

$$\frac{d^2 r}{dt^2} = -\frac{mMG}{r^2} + \frac{L^2}{mr^3} \quad -(4)$$

where a mass m orbits a mass M, and where G is Newton's constant. Eq. (3) defined the Bohr radii:

$$r = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \quad -(5)$$

upon quantization as follows:

$$L = n\hbar. \quad -(6)$$

Here n is the principal quantum number and $\hbar$ is the reduced Planck constant. The Bohr radius is defined for the lowest value of n, which is unity. In general:

$$n = 1, 2, 3, \ldots \quad -(7)$$

The total orbital linear velocity of the electron in the Bohr atom is {1 - 11}:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(8)$$

In view of Eq. (2):

$$v = v_\theta = r\frac{d\theta}{dt} = r\omega \quad -(9)$$

where the angular velocity ω is the spin connection of x theory {1 - 10}, which is based on Cartan geometry with non zero torsion. The Bohr velocity is therefore:

$$v = v_\theta = \omega r = \frac{L}{mr} = \frac{n\hbar}{mr} \quad - (10)$$

where r is the Bohr radius (5). The velocity v is defined in special relativity by the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = (c^2 - v^2)\, dt^2 \quad - (11)$$

where τ is the proper time. Therefore v used in special relativity is the classically defined v of Eq. (8), and so v in the Sommerfeld atom is the same as the Bohr velocity. This realization is a useful way of relating the two theories.

The hamiltonian of the Bohr atom is therefore:

$$H = \frac{1}{2} m v^2 - \frac{e^2}{4\pi\epsilon_0 r} = T + V \quad - (12)$$

in which the velocity is:

$$v = v_\theta = \frac{L}{mr}. \quad - (13)$$

Here T is the classical kinetic energy, V is the classical potential energy and E the total energy is a constant of motion defined by:

$$H = E = \frac{L^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}. \quad - (14)$$

From Eq. (3):

$$\frac{L^2}{2mr^2} = \frac{e^2}{8\pi\epsilon_0 r} \quad - (15)$$

so the total energy is

$$E = \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{e^2}{8\pi\epsilon_0 r} \quad -(16)$$

and is negative valued because it is an energy of attraction of the electron to the proton. From Eqs. (5) and (16):

$$E = \frac{-me^4}{32\pi^2\epsilon_0^2 n^2 \hbar^2} \quad -(17)$$

which are the non relativistic energy levels of the Bohr atom, Q. E. D. For atomic hydrogen (H) they are also the energy levels of the Schroedinger equation for H.

The orbit of the Bohr theory (1) and the Leibniz theory (4) is an ellipse. One theory is transformed into another as follows:

$$k = m\underline{M}G \rightarrow \frac{e^2}{4\pi\epsilon_0} \quad -(18).$$

The ellipse is defined by:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(19)$$

where the half right latitude is:

$$\alpha = \frac{L^2}{mk} \quad -(20)$$

and where the ellipticity is (1 - 11}:

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(21).$$

The semi major axis is:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2E} \quad -(22)$$

and the semi minor axis is:

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2mE)^{1/2}} \quad -(23)$$

The perihelion is the distance of closest approach of m to M and is defined by:

$$r_{min} = a(1-\epsilon) = \frac{\alpha}{1+\epsilon} \quad -(24)$$

and the aphelion is the maximum separation of m and M, defined by:

$$r_{max} = a(1+\epsilon) = \frac{\alpha}{1-\epsilon} \,. \quad -(25)$$

The Bohr radius is given immediately by the half right latitude:

$$\alpha = r = \frac{L^2}{mk} = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \,. \quad -(26)$$

Bohr in 1913 effectively assumed that the ellipse reduces to a circle, in which case the eccentricity vanishes:

$$\epsilon = 0 \,. \quad -(27)$$

From Eq. (21) this assumption means that:

$$\frac{2EL^2}{mk^2} = -1 \quad -(28)$$

so the energy levels of the Bohr atom are given immediately by Eq. (28):

$$E = -\frac{me^4}{32\pi^2\epsilon_0^2 n^2 \hbar^2} \quad -(29)$$

which is Eq. (17), Q. E. D. This result gave the main features of the spectrum of atomic H to state of art experimental precision. The fundamental reason for this is now known, the theory is an example of Cartan geometry and ECE theory. The way that the Bohr theory is taught however is that Bohr assumed that the force of attraction is equal to the centripetal force. This assumption is Eq. (3), and is equivalent to assuming a circular orbit. In the Schroedinger theory of H on the other hand it is not assumed in general that the Coulombic force of attraction is equal to the centripetal force, yet the same energy levels of H emerge from both theories. The Schroedinger atom is an example of the ECE wave equation {1 - 10} in the non relativistic limit of the fermion equation, and is again the result of geometry. The reason why the H energy levels are the same in the Bohr and Schroedinger theories of H is that the H energy levels in the Schroedinger atom are S orbitals. More generally the Bohr theory is a quantized ellipse, and the Bohr orbitals are no longer circles. The Bohr orbitals are developed in general into Schroedinger orbitals.

The Sommerfeld theory of the atom of 1915 was the first relativistic quantum theory and is an example of x theory in which the orbital is a precessing ellipse. It is based on the Hamiltonian:

$$H = E = (\gamma - 1)mc^2 - \frac{k}{r} \quad -(30)$$

where the relativistic kinetic energy is:

$$T = (\gamma - 1)mc^2 \quad -(31)$$

and where γ is the Lorentz factor:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(32)$$

As argued already the orbital velocity of the Sommerfeld theory is the same as that of the Bohr theory because the Sommerfeld theory is based on the Minkowski metric:

$$c^2 d\tau^2 = (c^2 - v^2)dt^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(33)$$

So the velocity of the Lorentz factor is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(34)$$

In the Bohr atom this is the Bohr velocity:

$$\frac{v}{c} = \frac{\alpha_f}{n} \quad -(35)$$

where the fine structure constant is defined by:

$$\alpha_f = \frac{e^2}{4\pi\hbar c\epsilon_0} = 0.00729735 1 \quad -(36)$$

and where the speed of light in vacuo is c.

Therefore the Lorentz factor of the Sommerfeld theory of the atom is:

$$\gamma = \frac{dt}{d\tau} = \left(1 - \left(\frac{\alpha_f}{n}\right)^2\right)^{-1/2} \quad -(37)$$

In addition to the energy equation (30), the force equation of the Sommerfeld atom must also be considered. The relativistic force corresponding to the relativistic kinetic energy used in Eq. (30) is defined {11} by:

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(\gamma m\underline{v}) \quad -(38)$$

so the work integral produces the result:

$$\int \underline{F} \cdot d\underline{r} = (\gamma - 1)mc^2 \quad -(39)$$

In Eq. (38) the relativistic linear momentum is:

$$\underline{p} = \gamma m \underline{v} . \quad -(40)$$

The relativistic force is therefore:

$$\underline{F} = \gamma m \frac{d\underline{v}}{dt} + m \underline{v} \frac{d\gamma}{dt} \quad -(41)$$

where

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv} \frac{dv}{dt} . \quad -(42)$$

Therefore:

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2} \quad -(43)$$

and the relativistic force is defined by:

$$\underline{F} = \gamma m \frac{d\underline{v}}{dt} + m \gamma^3 \frac{v}{c^2} \underline{v} . \quad -(44)$$

If it is assumed that:

$$v \ll c \quad -(45)$$

the relativistic force is approximated by:

$$\underline{F} \sim \gamma m \frac{d\underline{v}}{dt} . \quad -(46)$$

In plane polar coordinates {1 - 11} the Sommerfeld force is therefore:

$$F = \gamma m (\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \quad -(47)$$

where k is defined by:

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(48)$$

It follows from Eq. (47) that:

$$m\frac{d^2 r}{dt^2} = \frac{L^2}{mr^3} - \left(1 - \left(\frac{d_j}{n}\right)^2\right)^{1/2} \frac{e^2}{4\pi\epsilon_0 r^2} \quad -(49)$$

and is not zero in general. It differs from the Bohr orbit (1) through the Lorentz factor multiplying the second term on the right hand side.

The most general type of conical section of x theory {1 - 10} is:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(50)$$

and the force law of the Sommerfeld orbit is defined in general by:

$$F = \gamma m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\gamma L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = -\frac{k}{r^2} \quad -(51)$$

in which:

$$m\ddot{r} = m\frac{d^2 r}{dt^2} = -\frac{L^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \quad -(52)$$

From Eqs. (50) and (52):

$$m\frac{d^2 r}{dt^2} = \frac{x^2 L^2}{mr^2}\left(\frac{1}{r} - \frac{1}{\alpha}\right) \quad -(53)$$

In a circle:

$$r = \alpha \quad -(54)$$

so for a circular orbit:

$$m\frac{d^2r}{dt^2} = 0 \quad -(55)$$

which is the Bohr theory. In the Sommerfeld theory with its precessing elliptical orbitals (50), Eq. (53) is true for all x.

From Eq. (51):

$$m\frac{d^2r}{dt^2} = \frac{L^2}{mr^3} - \frac{1}{\gamma}\frac{k}{r^2} \quad -(56)$$

and from Eqs. (53) and (56):

$$x^2 = \left(\frac{L^2}{mr^3} - \frac{1}{\gamma}\frac{k}{r^2}\right)\left(\frac{L^2}{mr^3} - \frac{L^2}{\alpha m r^2}\right)^{-1} \quad -(57)$$

The precessing ellipse reduces to a static ellipse if:

$$x = 1 \quad -(58)$$

in which case:

$$\frac{1}{\gamma}\frac{k}{r^2} = \frac{L^2}{\alpha m r^2} \quad -(59)$$

and

$$\alpha = \frac{\gamma L^2}{mk} = \gamma r_B \quad -(60)$$

where r_B is the Bohr radius:

$$r_B = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \quad -(61)$$

For a static ellipse, Eq. (53) reduces to:

$$m\ddot{r} = \frac{L^2}{mr^3} - \left(1-\left(\frac{\alpha_f}{n}\right)^2\right)^{1/2}\frac{e^2}{4\pi\epsilon_0 r^2}$$

$$= \frac{e^2}{4\pi\epsilon_0 r^2}\left(1-\left(1-\left(\frac{\alpha_f}{n}\right)^2\right)^{1/2}\right) \quad -(62)$$

which is Eq. (49) Q. E. D, using the Bohr radius (61). In the Bohr theory:

$$r_B = \alpha \quad -(63)$$

and the static ellipse reduced to a circle. Recall that the Bohr and Sommerfeld velocities are the same, and that the Bohr velocity is defined by Eq. (10). Using this condition in Eq. (57) results in an expression for x:

$$x^2 = \left(1-\frac{1}{\gamma}\right)\left(1-\frac{r}{\alpha}\right)^{-1} \quad -(64)$$

where r in this expression is the Bohr radius:

$$r = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2} = \frac{\hbar}{mc}\cdot\frac{n^2}{\alpha_f} \quad -(65)$$

and where the Lorentz factor is defined by Eq. (37).

In summary the Sommerfeld atom is defined by a precessing ellipse of x theory in which:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(66)$$

and where the half right latitude is:

$$\alpha = r\left(1-\frac{1}{x^2}\left(1-\frac{1}{\gamma}\right)\right)^{-1} \quad -(67)$$

in which r is the Bohr radius:

$$r = \frac{\hbar}{mc} \cdot \frac{n^2}{\alpha_f} \quad -(68)$$

The energy levels of the Sommerfeld atom are:

$$E = mc^2\left(\left(1-\left(\frac{\alpha_f}{n}\right)^2\right)^{-1/2} - 1 - \left(\frac{\alpha_f}{n}\right)^2\right). \quad -(69)$$

The ellipticity of the Sommerfeld atom is:

$$\epsilon^2 = 1 + \frac{2E\alpha}{k}. \quad -(70)$$

By choosing x in a given range these properties can be computed, and the results of this computational analysis are given in Section 3. In Note 266(4) accompanying UFT266 on www.aias.us a new type of quantization of orbital theory has been suggested. This is named Eckardt quantization to distinguish it from Bohr and Sommerfeld quantization. Eckardt quantization is based on the theory of orbits {1 - 11} in which the precession factor can be expressed as:

$$x = 1 + \frac{r_0}{\alpha} \quad -(71)$$

where r_0 is a constant. In precessing planetary orbits, observation to state of art precision shows this to be:

$$r_0 = \frac{3MG}{c^2}. \quad -(72)$$

In atomic theory however r_0 can be assumed to be an unknown constant for the sake of argument. Eckardt quantization assumes that x is the principal quantum number:

$$x = n \quad -(73)$$

which means that:

$$\alpha = \frac{r_0}{n-1}, \quad n \neq 1. \quad -(74)$$

This theory results in waves being superimposed on the ellipse as illustrated in Section 3. The ellipse no longer precesses but develops these waves, related to de Broglie / Compton wavelengths. There are n waves per ellipse. In future work attempts can be made to relate the Eckardt and Bohr quantization schemes because it is known in general that the Bohr orbital is also an ellipse in which the half right latitude and eccentricity is also quantized.

3. COMPUTATION AND DISCUSSION

By Dr. Horst Eckardt

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Orbitals of the Bohr and Sommerfeld atoms with quantized x theory

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3 Computation and discussion

The first graph (Fig. 1) shows the Bohr circular orbits for $n = 1$ to $n = 4$ in atomic units:

$$\alpha_f = 0.0072973525, \quad c = \frac{1}{\alpha_f}, \quad \hbar = m = k = 1. \tag{75}$$

Then Eq.(5) simply reads

$$r_B = n^2. \tag{76}$$

The corresponding Bohr energy levels (Eq.(17)) are

$$E_B = \frac{1}{2\, n^2} \tag{77}$$

in Hartree units. The Sommerfeld energy is given by Eq.(69). The Bohr and Sommerfeld energies are shown in Table 1, together with the γ factors, for quantum numbers n. The differences in energy are small and the γ factors deviate from unity by less than 10^{-4}. The deviations become even smaller for growing n.

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n	E_{Bohr}	E_{Somm}	γ
1	-0.5000000	-0.4999800	1.0001065
2	-0.1250000	-0.1249988	1.0000266
3	-0.0555556	-0.0555553	1.0000118
4	-0.0312500	-0.0312499	1.0000067

Table 1: Bohr and Sommerfeld energy levels (in Hartree units) and γ factor for quantum numbers n.

The precession factor of the Sommerfeld theory x^2 is related to the half-right latitude α of the precessing ellipse by Eqs.(64) and (67). x^2 depends on the quantum number n via the Bohr radius (75) and the velocity (35) appearing in the γ factor (37). The dependence $x^2(\alpha)$ has been graphed in Fig. 2 as a function of the argument $\alpha \cdot n^2$ so that all Bohr radii are shifted to $\alpha = 1$ and can be compared directly. There is a sharp pole at $\alpha \approx r_B$ or, more precisely, $\alpha = \gamma r_B$. This pole becomes even sharper for increasing n. Therefore there is only a very small range around the Bohr radii where the Sommerfeld ellipses are defined, namely in the region with $x^2 \approx 1$. We see that the Bohr quantization of orbits is relaxed slightly in Sommerfeld theory but essentially remains valid.

The inverse relation $\alpha(x)$ is graphed in Fig 4. the α values have been normalized by r_B or n^2, respectively, so that the y scale is comparable. We see a very small variation of the α range which gets even smaller for rising x, in accordance with Fig. 3.

The ellipticity ϵ of Sommerfeld theory can be expressed by means of Eqs.(20) and (21) by

$$\epsilon = \sqrt{1 + 2\frac{E\,\alpha}{k}} \tag{78}$$

(see Eq.(70)) with α given by Eq.(67) and E given by Eq.(69). Thus, ϵ depends on x and n. Fig. 4 shows that ϵ is undefined for $x < 0.8$ since the square root argument is negative there. For growing x, the ellipticity is bound by an asymptote for each value of n. The small magnitude of ϵ shows again that Sommerfeld ellipses are extremely close to circles.

The effect of the Bohr velocity (35) has been investigated by artificially doubling this term. then $\epsilon(x)$ only starts at 1.6 (Fig. 5). This is unphysical because $x = 1$ must be included in the range of ϵ. The results are sensitive to the value of v.

The last part of this section investigates the Eckardt quantization. Inserting Eq.(74) into the equation for the precessing ellipse (50) gives

$$r = \frac{r_0}{(n-1)\,\epsilon\cos(n\theta)}. \tag{79}$$

These orbits are shown in Fig. 6. For $n = 1$ where r diverges a circular orbit has been assumed and a constant $\epsilon = 0.3$ has been used for all graphs. It can be seen that Eckardt quantization gives closed orbits (standing circular waves) with n being the number of maxima.

From Sommerfeld theory we know that the ellipticity is a function of energy, therefore we try to derive a corresponding expression for Eckardt quantization. Since the orbits in Eckardt quantization are highly elliptic, we have to use both components of the velocity. According to earlier papers we have

$$v_r = \frac{\epsilon\,L}{\alpha\,m}\sin(n\,\theta), \tag{80}$$

$$v_\theta = \frac{L}{m\,r}. \tag{81}$$

By means of

$$\cos(n\,\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right), \tag{82}$$

$$\sin(n\,\theta) = \sqrt{1 - \cos^2(n\,\theta)} \tag{83}$$

we obtain for the squared modulus of velocity after some arithmetics

$$v^2 = \frac{\left(\left(\epsilon^2 - 1\right) \, r + 2\,\alpha\right) \, L^2}{\alpha^2 \, m^2 \, r}. \tag{84}$$

The total energy is

$$E = \frac{1}{2} m v^2 - \frac{k}{r} \tag{85}$$
$$= \frac{1}{r} \left(\frac{L^2}{\alpha \, m} - k \right) + \left(\epsilon^2 - 1\right) \frac{L^2}{2\,\alpha^2\,m}. \tag{86}$$

This expression must be constant, therefore the r dependence must vanish. This is ensured by setting

$$L^2 = k\,\alpha\,m \tag{87}$$

which relates the angular momentum with a certain half-right latitude. Then the energy becomes

$$E = \frac{\left(\epsilon^2 - 1\right) \, k}{2\,\alpha} \tag{88}$$

from which follows

$$\epsilon^2 = 1 + \frac{2\,E\,\alpha}{k}. \tag{89}$$

Astonishingly this is the same expression as Eq.(70) or (78) derived from Sommerfeld theory. However, the energy is not quantized, α is quantized according to Eq.(74):

$$\alpha = \frac{r_0}{n-1}, \quad n \neq 1. \tag{90}$$

Therefore the orbits look different to Sommerfeld theory, and ϵ is not small for $n > 1$. The first five orbits are graphed in Fig. 7 and the corresponding ϵ values are shown there. Compared to Fig. 6, the orbits do not shrink with n but keep their maximum radius.

The energy could be quantized by inseting the quantized α into Eq.(88) but then E depends on n instead of $1/n^2$ as in Bohr and Sommerfeld theory. A correct behaviour of $E(n)$ would require additional quantization constraints for ϵ.

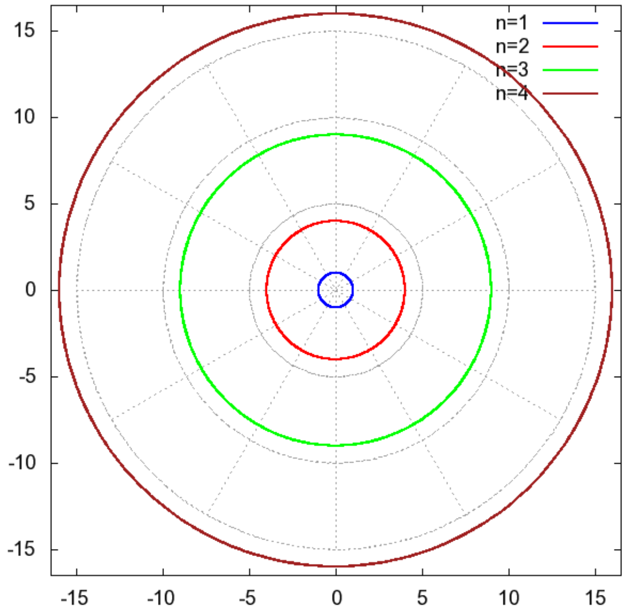

Figure 1: Bohr radii $r_B = n^2$ for quantum numbers n.

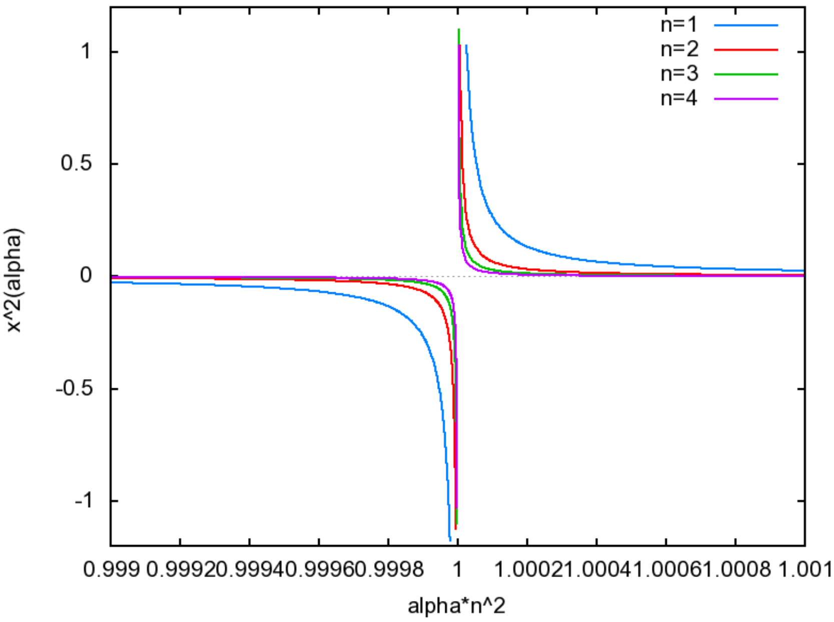

Figure 2: Sommerfeld precession factor $x^2(\alpha \cdot n^2)$ for quantum numbers n.

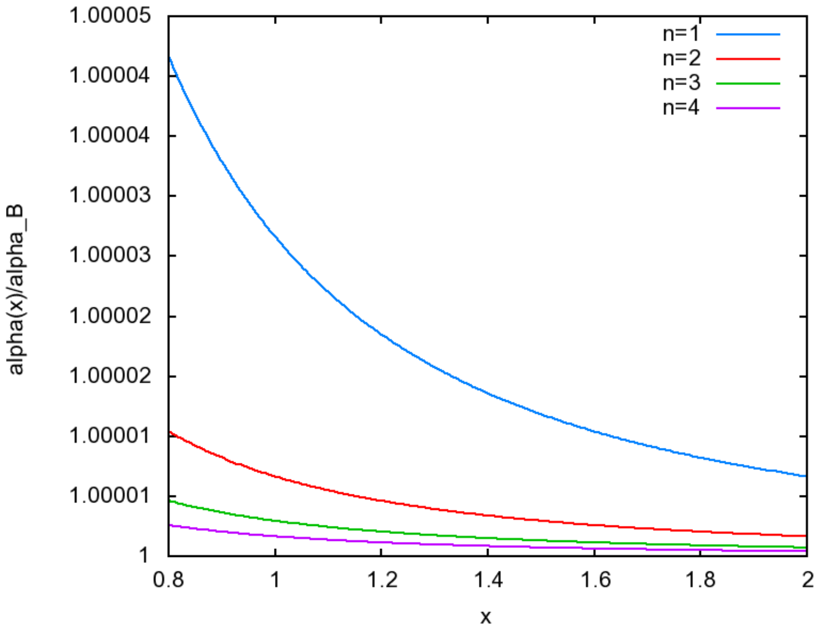

Figure 3: Normalized half-right latitude $\alpha(x)/n^2$ for Sommerfeld ellipses.

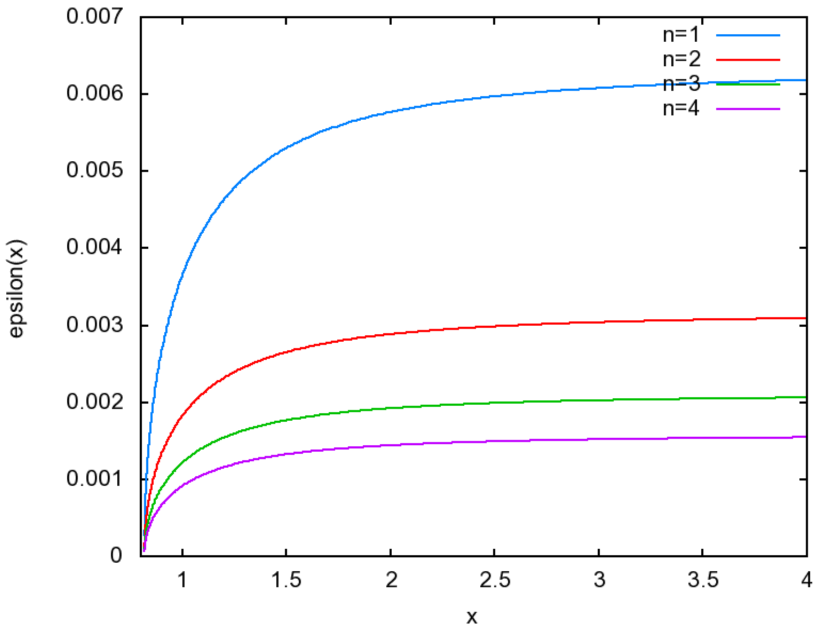

Figure 4: Ellipticity $\epsilon(x)$ for Sommerfeld ellipses.

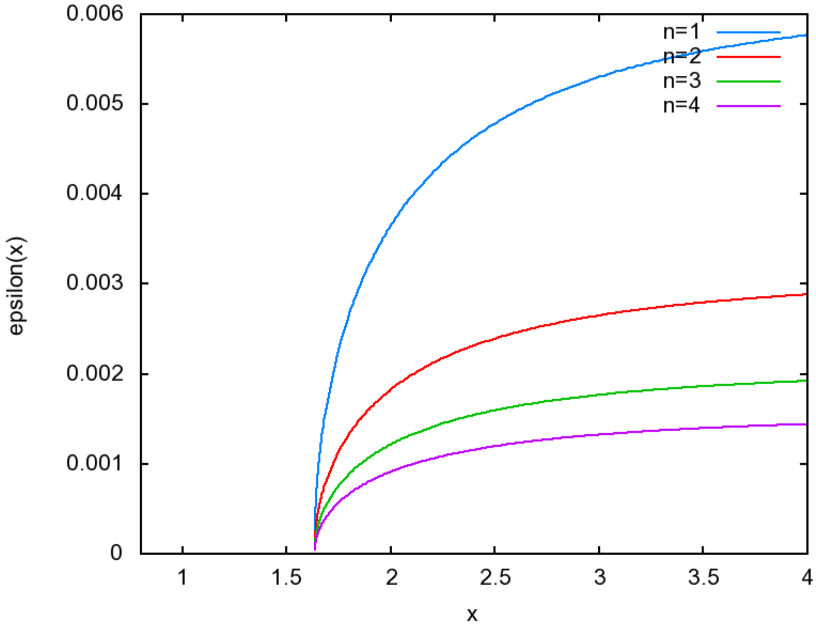

Figure 5: Ellipticity $\epsilon(x)$ for Sommerfeld ellipses with artificially enhanced orbital velocity $v \to 2v$.

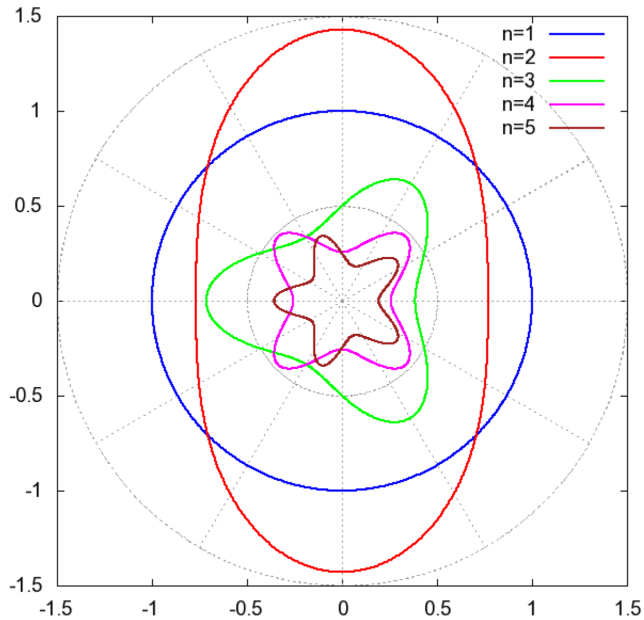

Figure 6: Orbitals of Eckardt quantization with $\epsilon = 0.3$, $r_0 = 1$.

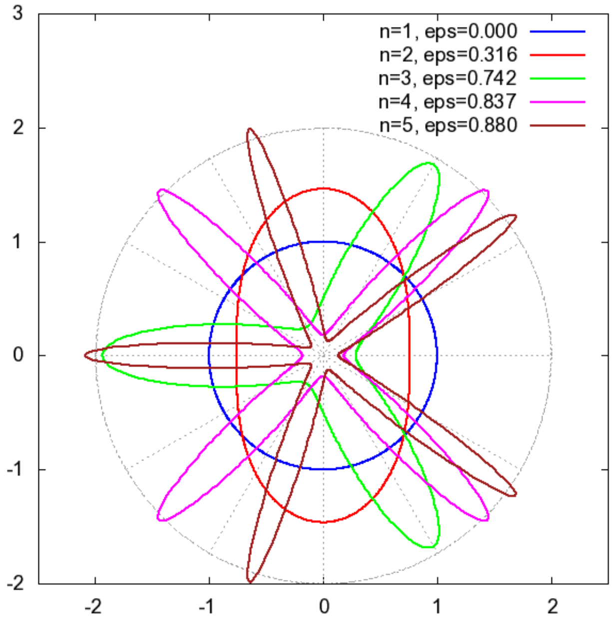

Figure 7: Orbitals of Eckardt quantization with variable ϵ with $r_0 = 1$, $E = -0.45$.