EXPECTATION VALUES OF ELLIPTICAL FUNCTIONS OF x THEORY:

APPLICATION TO THE H ATOM.

by

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ABSTRACT

It is shown that in general, a three dimensional orbit can be expressed as two elliptical functions or planar orbits. In classical gravitational theory and electrostatics the elliptical function describes the planar orbit due to an inverse square law of attraction. These functions are quantized to give a novel description of expectation values of the H atom and a great deal of new information. This method can be extended to any material with contemporary methods of computational quantum chemistry. The effect of ubiquitous Thomas precession is considered on the H atom and another set of novel results obtained.

Keywords: ECE theory, x theory, three dimensional orbits, quantization of elliptical functions, ubiquitous Thomas precession.

UFT 267

1. INTRODUCTION

In the immediately preceding papers of this series {1 - 10} the x theory of orbits has been developed in gravitational theory. Here x is the experimentally observed precession factor of any planar orbit in astronomy. This planar orbit is one where a mass m orbits a mass M in a precessing ellipse. The x theory uses plane polar coordinates and is based directly on the experimental data. The precessing elliptical function thus obtained has been used to give a straightforward explanation of precisely observable data in astronomy. To date the following phenomena have been described with x theory to state of art experimental precision: orbital precession, electromagnetic deflection due to gravitation, gravitational time delay, relativistic photon velocity, photon mass and the gravitational red shift. In a broader context ECE theory has been applied to the cosmological red shift in UFT 49 on www.aias.us. The x theory has also provided a definitive refutation of Einsteinian general relativity by showing that the latter develops an infinity when it tries to describe orbital precession self consistently and when it is compared with the precisely correct x theory. The origin of the x factor has been shown to be the ubiquitous Thomas precession, the rotation of the Minkowski metric at a constant angular velocity. The latter has been shown to be the spin connection of ECE theory, which is based directly on Cartan geometry. Furthermore, the velocity curve and hyperbolic orbits of a spiral galaxy have been shown to be due to the underlying ECE theory. In this case the Einstein theory fails catastrophically as is well known. While the observed velocity curve goes to a plateau, the Einstein theory goes to zero. It is also well known to scholarship that the Einstein theory is riddled with mathematical errors, primary among which is the neglect of Cartan torsion. So physics has split within the last decade into obsolete and almost completely obscure dogma (the standard model), and the Baconian ECE theory which is a return to the enlightened scientific principles of several hundred years. In the immediately preceding paper the x theory was used to forge a new quantum mechanics, exemplified by application to the Bohr and Sommerfeld theories of the atom in the old quantum theory. The Bohr theory was shown to be the circular limit of the elliptical function of x theory that emerges from the classical treatment of an electron orbiting a proton in the hydrogen atom. The Bohr radius was shown to be the half right latitude and the Bohr energy levels of the H atom shown to be the immediate result of vanishing ellipticity. The elliptical function was shown to be a necessary and sufficient description of atomic spectra in the non-relativistic old quantum theory. The Sommerfeld theory of the atom was the first relativistic quantum theory, and was shown to be due to a precessing elliptical function in which x is no longer unity.

It is well known that the energy levels of the Bohr atom in atomic H are the same as the energy levels of atomic hydrogen in the Schroedinger theory, modern quantum mechanics. In order to develop x theory to include the Schroedinger type of quantization it is first necessary to consider three dimensional orbits. This is because orbitals are three dimensional while orbits are almost always planar. In Section 2 it is shown that any three dimensional orbit can be considered in terms of two planar ellipses. Each of these ellipses can be quantized using the Schroedinger method to produce two sets of expectation values. This is a wholly original procedure that gives a novel characterization of the H atom and in consequence all materials using the methods of computational quantum mechanics. These are very highly developed as is well known. The H atom is the simplest atom and some of the calculations can be carried out analytically. The effect of the ubiquitous Thomas precession on these novel data sets is considered in the simplest possible way.

In Section 3, computer algebra is applied to obtain sets of expectation values for both types of elliptical function. In general the two elliptical functions give completely different results and the results are tabulated and discussed. The Thomas precession produces interesting relativistic corrections which in future work can be compared with results from the fermion equation. The latter improves the Dirac equation by removing unphysical negative energy levels. Both the fermion equation and x theory are obtained from the geometrical principles upon which ECE is based directly: the well known principles of Cartan geometry.

2. QUANTIZATION OF THREE DIMENSIONAL ORBITS INTO ORBITALS OF THE H ATOM.

Consider any orbit in three dimensions. The lagrangian in spherical polar coordinates is:

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{k}{r} = T - V \quad -(1)$$

where k is a constant and where the kinetic energy is:

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) \quad -(2)$$

where m is the orbiting mass. The three Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(3)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(4)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(5)$$

From Eq. (3):

$$m \left(\ddot{r} - r \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right) \right) = -\frac{k}{r^2} \quad -(6)$$

From Eq. (4):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt} m r^2 \dot{\theta} = 0 \quad -(7)$$

and from Eq. (5):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt} m r^2 \sin^2\theta \, \dot{\phi} = 0 \quad -(8)$$

so the conserved angular momenta are:

$$L_1 = m r^2 \dot{\theta} \quad -(9)$$

and:

$$L_2 = m r^2 \sin^2\theta \, \dot{\phi}. \quad -(10)$$

It follows that:

$$m\frac{d^2 r}{dt^2} = -\frac{k}{r^2} + \frac{L^2}{m r^3} \quad -(11)$$

where:

$$L^2 = L_1^2 + L_2^2. \quad -(12)$$

The format of the Leibniz equation (11) is not changed by going from two to three dimensions, but the total angular momentum is defined in three dimensions by Eq. (12).

It follows that:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{-m}{L_1 \dot{\theta}} \ddot{r} \quad -(13)$$

and:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = -\frac{m}{L_2 \dot{\phi}} \ddot{r} \quad -(14)$$

so there are two Binet equations:

$$-\frac{L_1^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{L^2}{m^2 r^3} = -\frac{k}{r} \quad -(15)$$

and:

$$-\frac{L_2^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) - \frac{L^2}{m^2 r^3} = -\frac{k}{r} \quad -(16)$$

in which:

$$m\frac{d^2 r}{dt^2} = -\frac{L_1^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{L_2^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right). \quad -(17)$$

The Leibniz equation is given via the Binet equations by two ellipses:

$$r = \frac{\alpha_1}{1+\epsilon_1 \cos\theta} \quad -(18)$$

and

$$r = \frac{\alpha_2}{1+\epsilon_2 \cos\phi} \quad -(19)$$

giving:

$$m\frac{d^2 r}{dt^2} = \frac{L_1^2}{mr^2}\left(\frac{1}{r} - \frac{1}{\alpha_1}\right) = \frac{L_2^2}{mr^2}\left(\frac{1}{r} - \frac{1}{\alpha_2}\right). \quad -(20)$$

Therefore a three dimensional orbit can always be analyzed with two planar elliptical orbits of x theory, with x = 1.

When considering the hydrogen atom both ellipses can be quantized and their expectation values computed. For example if the starting classical hamiltonian is defined by the plane polar coordinates (r , ϕ) then:

$$H = E = \frac{p^2}{2m} - \frac{k}{r} \quad -(21)$$

where the classical kinetic energy is:

$$T = \frac{p^2}{2m} = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right) \quad -(22)$$

and where the classical potential energy is:

$$V = -\frac{k}{r} \quad -(23)$$

The solution of Eq. (21) is the ellipse or more generally the conical section {11}:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} . \quad -(24)$$

For the ellipse the half right latitude is:

$$\alpha = \frac{L^2}{mk} \quad -(25)$$

and the ellipticity is:

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2} \quad -(26)$$

where E is the conserved total energy and where L is the conserved total angular momentum:

$$L = m r^2 \omega \quad -(27)$$

where

$$\omega = \frac{d\phi}{dt} = \frac{L}{mr^2} \quad -(28)$$

is the spin connection or angular velocity {1 - 10}.

The Schroedinger equation is obtained directly from the classical equation (21) using:

$$p^2 \phi = -\hbar^2 \nabla^2 \phi \quad -(29)$$

where $\hbar$ is the reduced Planck constant and where ϕ is the wavefunction. So the Schroedinger equation is

$$-\frac{\hbar^2}{2m} \nabla^2 \phi = \left(E + \frac{k}{r} \right) \phi \quad -(30)$$

and this is a quantized Leibniz equation. The ellipse (24) is the solution of the classical equation so it must also be a solution of the quantized equation by the correspondence principle that quantized solutions reduce in well defined limits to their classical counterparts. There are exceptions to this principle, in that some quantum phenomena such as zero point energy do not have a classical counterpart. It follows that a great deal of new information about the H atom can be obtained by calculating the expectation values <r>. This is true in general, a great deal of new information can be obtained in this way in general for any material using the highly developed methods of computational quantum chemistry. The expectation values thus obtained would characterize any given atom, molecule or material in an entirely new way, so new data banks can be built up. The basis for this development is x theory with x = 1.

The expectation values are:

$$\langle r \rangle = \int \phi^* \frac{\alpha}{1 + \epsilon \cos \phi} \phi \, d\tau \quad -(31)$$

In the Bohr theory of the atom:

$$\langle r \rangle = \int \phi^* r_B \phi \, d\tau = r_B \int \phi^* \phi \, d\tau \quad -(32)$$

in which the Born normalization is:

$$\int \phi^* \phi \, d\tau = 1 \quad -(33)$$

so:

$$\langle r \rangle = r_B . \quad -(34)$$

The Bohr theory (see UFT266 on www.aias.us) corresponds to:

$$\alpha = r_B , \quad \epsilon = 0 \quad -(35)$$

but in the Schroedinger atom the wavefunctions of H are {1 - 10}::

$$\phi = R_{nl}(r) Y_{lm}(\theta, \phi) \quad -(36)$$

where some values of $R_{nl}(r)$ are given in Note 267(3), and where the spherical harmonics Y are defined as in previous UFT papers. The integral of any function f over the volume element is defined by:

$$\int f \, d\tau = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_0^{\infty} f r^2 \sin\theta \, dr \, d\theta \, d\phi \quad -(37)$$

and the required definite integrals are:

$$\int_0^{\infty} r^n e^{-ar} dr = \frac{n!}{a^{n+1}} \quad -(38)$$

and:
$$\int_0^{\pi} \cos^n\theta \sin\theta \, d\theta = \frac{1 + (-1)^n}{n+1} \qquad -(39)$$

For example the 1S orbital of atomic hydrogen (H) is defined by:

$$n = 1, \; \ell = 0, \; m_\ell = 0 \qquad -(40)$$

so the wavefunction is:

$$\phi = R_{10} Y_{00} = 2\left(\frac{1}{r_B}\right)^{3/2} e^{-r/r_B} \cdot \frac{1}{2\pi^{1/2}} \qquad -(41)$$

The expectation value of the Bohr radius in this case is:

$$\langle r \rangle = \frac{1}{\pi r_B^3} \int_0^{2\pi} r_B \, d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 e^{-2r/r_B} \, dr = r_B \qquad -(42)$$

This simple calculation illustrates the fact that the normalization must be correctly defined, and the code written by Dr. Horst Eckardt used in this paper and in all previous UFT papers of this kind checks the normalization for each wavefunction. As shown in UFT266 the Bohr atom is obtained directly from the ellipse (24) in the limit of vanishing ellipticity. The ellipse in this limit gives the energy levels and Bohr radii directly, leading to a powerful new understanding of quantization. If the above calculation is repeated for the ellipse (24) the result is:

$$\langle r \rangle = \frac{1}{\pi r_B^2} \int_0^{2\pi} \frac{d\phi}{1 + \epsilon \cos\phi} \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 \exp\left(-\frac{2r}{r_B}\right) dr$$

$$= \frac{r_B}{2\pi} \int_0^{2\pi} \frac{d\phi}{1 + \epsilon \cos\phi} = \frac{r_B}{(1 - \epsilon^2)^{1/2}} \qquad -(43)$$

As discussed in Section 3 the same result is obtained for all the hydrogenic wavefunctions for the ellipse defined in the (r , ϕ) coordinates.

However, if the same procedure is applied to the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad - (44)$$

a completely different set of expectation values emerge. These two sets of expectation values can be used to characterize any material and are tabulated in Section 3. Note carefully that in the limit of vanishing ellipticity, the expectation value of the 1S orbital occurs at the Bohr radius, and that of the nS orbital at n multiplied by the Bohr radius.

The effect of ubiquitous Thomas precession is to rotate the Minkowski metric at a constant angular velocity ω . It appears on all scales and in all situations. In gravitational theory the velocity of the Thomas precession is defined by the equivalence principle applied to the rotational kinetic energy:

$$\frac{1}{2} m r^2 \left(\frac{d\phi}{dt}\right)^2 = \frac{1}{2} m v_\phi^2 = \frac{mMG}{r} \quad - (45)$$

so

$$v_\phi^2 = \frac{2MG}{r} . \quad - (46)$$

In electrostatics:

$$\frac{1}{2} m v_\phi^2 = \frac{e^2}{4\pi\epsilon_0 r} \quad - (47)$$

so the Thomas velocity is:

$$v_\phi^2 = \frac{e^2}{2\pi\epsilon_0 m r} \quad - (48)$$

As shown in UFT265 the rotation of the Minkowski metric causes all planar orbits to precess with the experimentally observed x factor, to give the precessing orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\phi)} \quad -(49)$$

where:

$$x = 1 + \frac{3mG}{c^2 \alpha}. \quad -(50)$$

Arguing in direct analogy:

$$mG \rightarrow \frac{e^2}{4\pi\epsilon_0 m} = \frac{\hbar c}{m} \alpha_f \quad -(51)$$

where α_f is the fine structure constant:

$$\alpha_f = \frac{e^2}{4\pi\hbar c \epsilon_0} = 0.00729351 \quad -(52)$$

and where the Compton wavelength is:

$$\lambda_c = \frac{h}{mc} = 2.426309 \times 10^{-12} \, m \quad -(53)$$

i. e.:

$$\frac{\hbar}{mc} = \frac{\lambda_c}{2\pi}. \quad -(54)$$

Therefore:

$$x = 1 + \frac{3mG}{c^2 \alpha} \rightarrow 1 + \frac{3\alpha_f \lambda_c}{2\pi \alpha} \quad -(54)$$

In the conventional notation of the spherical polar coordinates:

$$\begin{aligned} X &= r \sin\theta \cos\phi \\ Y &= r \cos\theta \sin\phi \\ Z &= r \cos\theta \end{aligned} \quad -(55)$$

so when:

$$\theta = \frac{\pi}{2} \quad -(56)$$

the plane polar coordinates are defined by:

$$X = r\cos\phi$$
$$Y = r\sin\phi \quad -(57)$$

In this case the relevant ellipse to consider is:

$$r = \frac{\alpha}{1 + \epsilon\cos\phi} \quad -(58)$$

and Thomas precession is generated by rotation about the Z axis so that the ellipse (58) becomes (49). Note carefully that in some textbooks in mathematics {12} the opposite notation is used as in Note 267(3), with ϕ and θ interchanged.

In both the classical gravitational and the classical electrostatic theory of precessing planar orbits:

$$H = E = T + V \quad -(59)$$

where:

$$T = \frac{1}{2} m v^2, \quad -(60)$$
$$V = -k/r. \quad -(61)$$

In gravitation:

$$k = mMG \quad -(62)$$

and in electrostatics:

$$k = \frac{e^2}{4\pi\epsilon_0}. \quad -(63)$$

The velocity in Eq. (60) is defined by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2 \quad -(64)$$

where:

$$\omega = \frac{d\phi}{dt} = \frac{L}{mr^2}. \quad -(65)$$

The force in both gravitation and electrostatics is:

$$F = m\left(\frac{d^2r}{dt^2} - \omega^2 r\right) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = -\frac{k}{r^2} \quad -(66)$$

and in both cases:

$$m\frac{d^2r}{dt^2} = -\frac{L^2}{mr^2}\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = -\frac{k}{r^2} + \frac{L^2}{mr^3}, \quad -(67)$$

if:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(68)$$

However, if:

$$r = \frac{\alpha}{1+\epsilon\cos(x\phi)} \quad -(69)$$

then:

$$m\frac{d^2r}{dt^2} = x^2\left(-\frac{k}{r^2} + \frac{L^2}{mr^3}\right) \quad -(70)$$

and the hamiltonian (59) changes from:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \left(E + \frac{k}{r}\right)\psi \quad -(71)$$

to:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \left(E + x^2\frac{k}{r} + (1-x^2)\frac{L^2}{2mr^2}\right)\psi$$

$$-(72)$$

Therefore in general the ubiquitous Thomas precession causes the Schroedinger equation to change from:

$$H\phi = E\phi = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{k}{r} \right)\phi \quad -(73)$$

to:

$$H\phi = E\phi = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{kx^2}{r} + \frac{(1-x^2)L^2}{2mr^2} \right)\phi \quad -(74)$$

The hamiltonian changes from:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{k}{r} \quad -(75)$$

to:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{x^2 k}{r} + \frac{(1-x^2)L^2}{2mr^2} \quad -(76)$$

and develops a new term in L^2. This term can be thought of as a relativistic correction. Since x does not depend on ellipticity, this correction is the same in the Bohr atom, where:

$$d(n=1) = r_B = 5.29177 \times 10^{-11} m \quad -(77)$$

Bohr type quantization produces:

$$L = n\hbar, \quad d = n^2 r_B \quad -(78)$$

where:

$$r_B = \frac{4\pi\epsilon_0\hbar^2}{me^2} \quad -(79)$$

is the Bohr radius. The entire structure of computational quantum chemistry is changed by the ubiquitous Thomas precession. In Eq. (76).

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3 Tabular and graphical work on expectation values

The Schroedinger equation for hydrogen-like orbitals with atomic number Z reads

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{Zk}{r} = E\psi. \tag{80}$$

With the product wave function approach (36) for ψ the angular part can be separated. The eigen values of the spherical harmonics $Y(\theta, \phi)$ for the angular Laplace operator Λ^2 in spherical coordinates are

$$\Lambda^2 Y(\theta, \phi) = -l(l+1)Y(\theta, \phi) \tag{81}$$

so that the spherical harmonics in Eq.(80) cancel out. An equation for the radial part only remains:

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR_{nl}) + \left(\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{Zk}{r}\right)R_{nl} = E\,R_{nl}. \tag{82}$$

Introducing the variable

$$u_{nl} = r\,R_{nl} \tag{83}$$

and dividing by R_{nl} leads to the equation

$$-\frac{\hbar^2}{2m}\frac{1}{u_{nl}}\frac{\partial^2 u_{nl}}{\partial r^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{Zk}{r} = E. \tag{84}$$

By re-introducing the radial part of the classical kinetic energy E_k and the classical angular momentum

$$L^2 = \hbar^2\,l(l+1) \tag{85}$$

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Eq.(84) can be written

$$E_k + \frac{L^2}{2mr^2} - \frac{Zk}{r} = E \tag{86}$$

with

$$E_k = \frac{1}{2}mv_r^2 = -\frac{\hbar^2}{2m}\frac{1}{u_{nl}}\frac{\partial^2 u_{nl}}{\partial r^2}. \tag{87}$$

It follows that v_r^2 can be expressed by the radial functions in the form

$$v_r^2 = -\frac{\hbar^2}{m^2}\frac{1}{u_{nl}}\frac{\partial^2 u_{nl}}{\partial r^2}. \tag{88}$$

This represents a classical electron velocity in the framework of quantum mechanics, a novel result. The velocities are dependent on the (n, l) quantum numbers and listed in Table 1. Besides a constant term, there are contributions of inverse powers of r, growing with quantum number l. The results are graphed in Fig. 1. For the s orbitals, velocities are highest near to the core, a result which is consistent with the probability distribution of wave functions. For p and d orbitals, velocities start at some distance from the core.

Next we consider the expectation values of the elliptic radius derived from hydrogenic orbitals, see Eq.(31). As explained in section 2, we have two possibilities of symmetrically placing the elliptic orbits in a 3D coordinate system: in the ϕ plane and in the θ plane. Placing the ellipse in the ϕ plane gives the result

$$\langle r \rangle = \frac{a_0}{\sqrt{1-\epsilon^2}} \tag{89}$$

with a_0 being the Bohr radius. This expectation value is the same for all quantum numbers (n, l, m_l). The reason is that the spherical harmonics depend on ϕ only by a phase factor $\exp(i\, m_l\, \phi)$ which is the same for all spherical harmonics.

Putting the ellipse into the θ plane gives quite different results. They now depend on the quantum numbers and give logarithmic expressions, independent of principal quantum number n, because the spherical harmonics do not depend on n. The results are presented in Table 2. The r expectation values are graphed in dependence of ellipticity ϵ which they depend on. In Fig. 2 the curves for $l = 0$ and $l = 1$ are shown, together with the result (89) for the ϕ orbit. We see that $\langle r \rangle$ for the ϕ orbit is very similar to that of the θ orbit for $l = 0$. The expectation values grow significantly for ϵ approaching unity, i.e. the ellipse passing into a parabola. In Fig. 3 this is shown for d orbitals. The effect is always more significant the lower the quantum number m_l is.

Finally we investigated $\langle r \rangle$ for precessing ellipses according to Eq.(74). The expectation values then contain expressions like

$$\int \frac{\sin(\theta)}{\cos(x\theta)+1}\, d\theta \tag{90}$$

which are not analytically integrable because the arguments of the trigonometric functions differ. Only for the ϕ ellipse an analytical result is obtained:

$$\langle r \rangle = \frac{a_0}{\pi\, x\, \sqrt{1-\epsilon^2}} \operatorname{atan}\left(\frac{\sqrt{1-\epsilon^2}\, \sin(2\,\pi\, x)}{(\epsilon+1)\, \cos(2\,\pi\, x) + \epsilon + 1} \right). \tag{91}$$

n	l	$v_r^2 \cdot m^2/\hbar^2$
1	0	$\frac{2\,Z}{a_0\,r} - \frac{Z^2}{a_0^2}$
2	0	$\frac{2\,Z}{a_0\,r} - \frac{Z^2}{4\,a_0^2}$
2	1	$-\frac{Z^2}{4\,a_0^2} + \frac{2\,Z}{a_0\,r} - \frac{2}{r^2}$
3	0	$\frac{2\,Z}{a_0\,r} - \frac{Z^2}{9\,a_0^2}$
3	1	$-\frac{Z^2}{9\,a_0^2} + \frac{2\,Z}{a_0\,r} - \frac{2}{r^2}$
3	2	$-\frac{Z^2}{9\,a_0^2} + \frac{2\,Z}{a_0\,r} - \frac{6}{r^2}$

Table 1: Squared orbital velocities v_r^2 in unts of $\hbar^2/m^2$ for different quantum numbers.

l	m	$\langle r \rangle$
0	0	$\frac{a_0}{2\,\epsilon}\,(\log(\epsilon+1) - \log(1-\epsilon))$
1	0	$\frac{3\,a_0}{2\,\epsilon^3}\,(\log(\epsilon+1) - 2\,\epsilon - \log(1-\epsilon))$
1	1	$\frac{3\,a_0}{4\,\epsilon^3}\,\left(\left(\epsilon^2-1\right)\,\log(\epsilon+1) + \log(1-\epsilon)\,\left(1-\epsilon^2\right) + 2\,\epsilon\right)$
2	0	$\frac{5\,a_0}{8\,\epsilon^5}\,\left(\epsilon^2-3\right)\,\left(\left(\epsilon^2-3\right)\,\log(\epsilon+1) + \log(1-\epsilon)\,\left(3-\epsilon^2\right) + 6\,\epsilon\right)$
2	1	$\frac{5\,a_0}{4\,\epsilon^5}\,\left(3\,\left(\epsilon^2-1\right)\,\log(\epsilon+1) - 4\,\epsilon^3 + 3\log(1-\epsilon)\,\left(1-\epsilon^2\right) + 6\,\epsilon\right)$
2	2	$\frac{5\,a_0}{16\,\epsilon^5}\,\left(\left(3\,\epsilon^4 - 6\,\epsilon^2 + 3\right)\,\log(\epsilon+1) + \log(1-\epsilon)\,\left(-3\,\epsilon^4 + 6\,\epsilon^2 - 3\right) + 10\,\epsilon^3 - 6\,\epsilon\right)$

Table 2: Expectation values of r for an ellipse in θ direction (X-Z plane).

For the θ orbits the integrals are not solvable analytically and must be evaluated numerically.

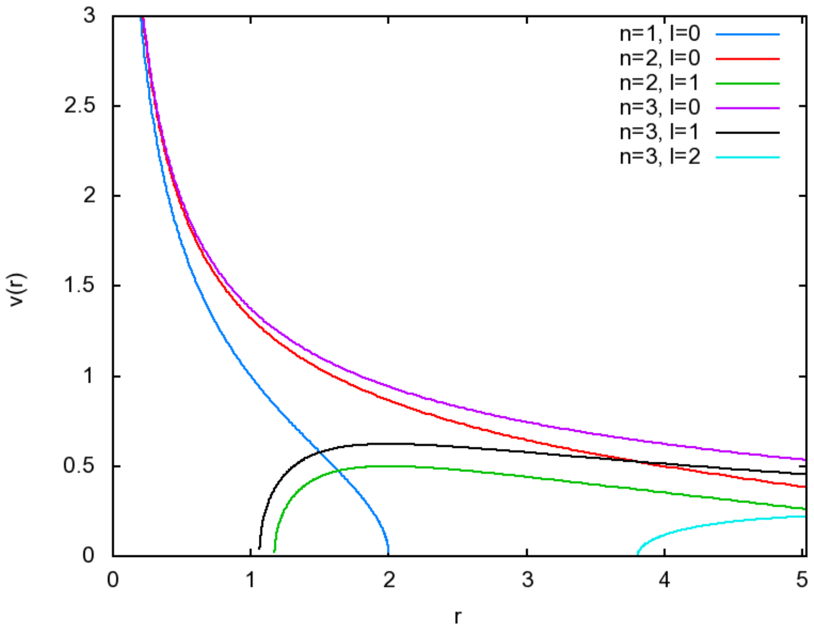

Figure 1: Velocity $v_r(r)$ for n and l quantum numbers.

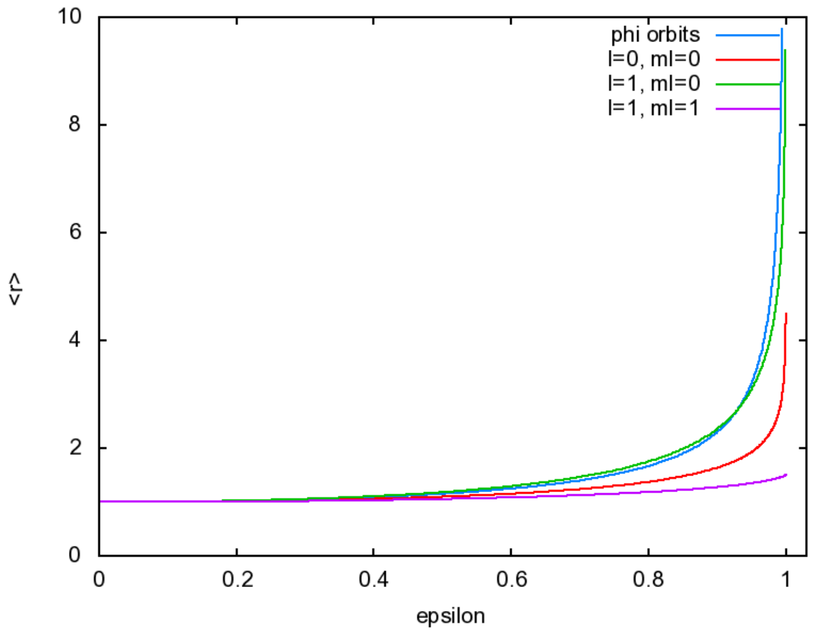

Figure 2: Expectation values of r for ϕ orbit and some θ orbits.

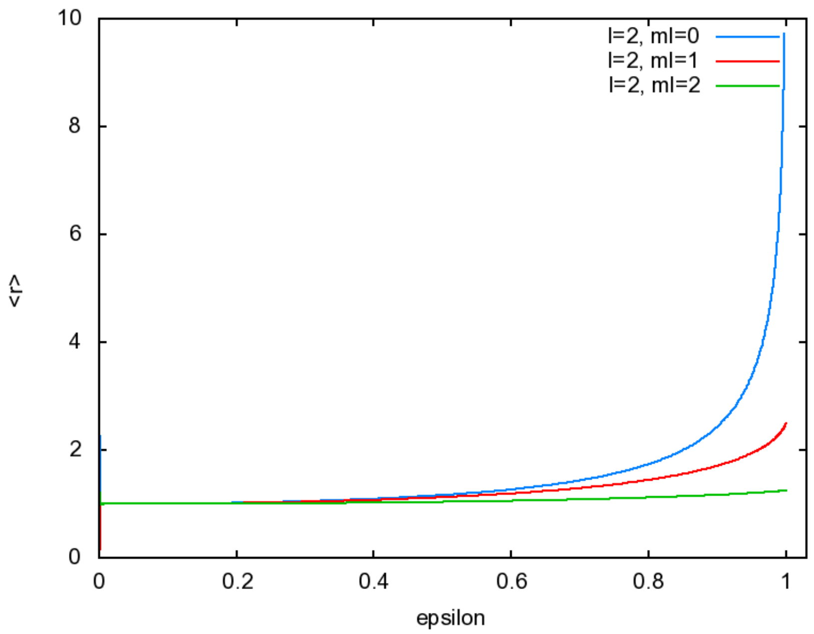

Figure 3: Expectation values of r for some more θ orbits ($l = 2$).

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