THE THEORY OF ORBITS IN SPHERICAL POLAR COORDINATES.

by

M. W. Evans and H. Eckardt,

Civil List, AIAS and UPITEC

(www.aias.us, www.atomicprecision.com, ww.et3m.net, www.upitec.org,

www.webarchive.org.uk)

ABSTRACT

The theory of orbits is developed using spherical polar coordinates and the inverse square law of attraction. In general the orbit is three dimensional and there are several constants of motion defined by the total angular momentum in three dimensions and by components. The orbit is an intricate function of the coordinates of the spherical polar coordinates and is not confined to a plane perpendicular to the axis of one angular momentum as in plane polar coordinates. There are several types of Coriolis and centripetal acceleration in general. A hamiltonian and lagrangian analysis is used to develop the theory.

Keywords: ECE theory, spherical polar coordinates, three dimensional orbits.

4FT 270

1. INTRODUCTION

In recent papers of this series {1 - 10} several fundamental phenomena have been analyzed in terms of a two dimensional conical section, notably the ellipse or precessing ellipse. In this paper the dynamics of the inverse square law of attraction are developed with spherical polar coordinates, producing the original result that orbits in general are three dimensional, not flat conical sections in a plane. This conclusion is the direct result of merging together two fundamental concepts, the inverse square law of attraction and the spherical polar coordinates, so the results are irrefutable mathematically. They are checked whenever possible by computer algebra. In Section 2 the orbit is developed rigorously without approximations and the various inter relations between angles given by integration using computer algebra. Fundamental concepts in spherical polar coordinates contain many more terms than in non inertial Newtonian dynamics and also more terms than dynamics developed in plane polar coordinates. In Section 3 the main results are graphed in three dimensions and animated. A discussion is given of the main features of the graphics. As usual this paper should be read with the background notes for UFT270 on www.aias.us.

2. ORBITAL THEORY

Consider the hamiltonian

$$H = E = \frac{1}{2} m v^2 - \frac{k}{r} \quad -(1)$$

in spherical polar coordinates. In gravitational theory m is a mass orbiting a mass M, and r is the distance between m and M. The constant k is mMG where G is Newton's constant. However the same theory applies to an electron orbiting a proton on the classical level, provided that radiation effects are neglected. In this case:

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(2)$$

where e is the charge on the proton and ϵ_0 the vacuum permittivity. The major new insights of this paper apply both to orbital theory and to classical electrodynamics. When Eq. (1) is quantized it gives the hydrogenic orbitals as is well known. The orbitals are intricate three dimensional functions defined by a product of the radial functions and the spherical harmonics. The whole of quantum mechanics is therefore based on the use of the laplacian in spherical polar coordinates. Curiously, orbit theory on the other hand has been confined for over four hundred years to plane polar coordinates, thus losing a great deal of information.

In spherical polar coordinates {11, 12}, the position vector is defined by:

$$\underline{r} = r\,\underline{e}_r \quad -(3)$$

and the unit vectors {11} by:

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \quad -(4)$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(5)$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(6)$$

in terms of the Cartesian unit vectors. There is a cyclically symmetric relation between the unit vectors:

$$\underline{e}_\phi \times \underline{e}_r = \underline{e}_\theta \quad -(7)$$

$$\underline{e}_\theta \times \underline{e}_\phi = \underline{e}_r \quad -(8)$$

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_\phi . \quad -(9)$$

The linear velocity is {11, 12}:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(10)$$

so $\underline{r}$ and $\underline{v}$ are not coplanar as in plane polar coordinates. The angular momentum is:

$$\underline{L} = m\underline{r}\times\underline{v} \quad -(11)$$

$$= mr^2\dot{\theta}\,\underline{e}_\phi - mr^2\dot{\phi}\sin\theta\,\underline{e}_\theta$$

and has two components. In plane polars it has only one component. Using Eqs. (11) in Eq. (4) – (6) it becomes clear that the angular momentum has three Cartesian components:

$$L_X = -mr^2\left(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi\right) -(12)$$

$$L_Y = mr^2\left(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi\right) -(13)$$

$$L_Z = mr^2\dot{\phi}\sin^2\theta. \quad -(14)$$

This result is the same as that derived in UFT269 in a different way. In planar orbital theory there is only one component. of angular momentum usually denoted:

$$\underline{L} = L_Z\,\underline{k}. \quad -(15)$$

In spherical polar coordinates however:

$$\underline{L} = L_X\,\underline{i} + L_Y\,\underline{j} + L_Z\,\underline{k} \quad -(16)$$

and conservation of angular momentum means that $\underline{L}$ and its three Cartesian components are all constants of motion:

$$\frac{d\underline{L}}{dt} = \frac{dL_X}{dt} = \frac{dL_Y}{dt} = \frac{dL_Z}{dt} = 0. \quad -(17)$$

The force in spherical polar coordinates is {11}:

$$\underline{F} = m\underline{a} = m(a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi) \quad -(18)$$

where:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 \quad -(19)$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 \quad -(20)$$

$$a_\phi = 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi} \quad -(21)$$

and contains many terms not present in plane polar coordinates or in a non inertial Newtonian development. These are developments of the Coriolis and centrifugal forces of planar orbits.

The square of the velocity is:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 \quad -(22)$$

so the hamiltonian is:

$$H = E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - \frac{k}{r} \quad -(23)$$

and the lagrangian is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + \frac{k}{r} \quad -(24)$$

These can be expressed in terms of the angle β, defined by:

$$\dot{\beta}^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 \quad -(25)$$

and the hamiltonian and lagrangian may be written as:

$$H = E = \frac{1}{2}m(\dot{r}^2 + \dot{\beta}^2 r^2) - \frac{k}{r} \quad -(26)$$

and
$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + \dot{\beta}^2 r^2 \right) + \frac{k}{r} \quad -(27)$$

The basics of the lagrangian theory are given in Note 270(7). It is possible to develop a two variable lagrangian theory consisting of the Euler Lagrange equations:
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(28)$$
and
$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad -(29)$$

and a three variable lagrangian theory consisting of the Euler Lagrange equation (28) together with:
$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(30)$$

and
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \,. \quad -(31)$$

Eq. (28) with the lagrangian (27) gives the force equation:
$$m \ddot{r} = m r \dot{\beta}^2 - \frac{k}{r^2} \quad -(32)$$

and Eq. (29) with the lagrangian (27) gives the angular velocity:
$$\dot{\beta} = \frac{L}{m r^2} \quad -(33)$$

in terms of the constant of motion L. Eqs. (32) and (33) give the three dimensional orbital equation:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(34)$$

which resembles a conic section, for example an ellipse. It is referred to for ease of reference as the beta ellipse or main orbit. The half right latitude and ellipticity are defined as follows in terms of the constants of motion E, the total energy, and L, the magnitude of the total angular momentum:

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2}. \quad -(35)$$

If the conic section is an ellipse, the semi major and semi minor axes are defined by:

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2E} \quad -(36)$$

and

$$b = \frac{\alpha}{(1-\epsilon^2)^{1/2}} = \frac{L}{(2mE)^{1/2}} \quad -(37)$$

From the lagrangian (24) and the Euler Lagrange equations (30) and (31):

$$\frac{dL_z}{dt} = \frac{d}{dt}\left(mr^2\dot{\phi}\sin^2\theta\right) = 0 \quad -(38)$$

and

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\sin\theta\cos\phi\,\dot{\phi}^2. \quad -(39)$$

Therefore the angular momentum:

$$L_z = mr^2\dot{\phi}\sin^2\theta \quad -(40)$$

is a constant of motion but the angular momentum:

$$L_1 = mr^2\dot{\theta} \quad -(41)$$

is not a constant of motion. Eqs. (38) and (14) give the same result from two entirely different methods, Q. E. D. The Z component of angular momentum is therefore a constant of motion, and in quantum mechanics it becomes the operator:

$$\hat{L}_Z \phi = m_\ell \hbar \phi \quad -(42)$$

where ϕ is the wavefunction, m_ℓ is a quantum number, and where $\hbar$ is the reduced Planck constant.

From Eqs. (12), (13) and (14) the square of the angular momentum is:

$$L^2 = m^2r^4\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta + \dot{\phi}^2 \sin^4\theta\right)$$

$$= m^2r^4\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \quad -(43)$$

so the hamiltonian (23) can be written as:

$$H = E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r} \quad -(44)$$

and the force equation (32) can be written as the Leibniz equation:

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{k}{r} \quad -(45)$$

The square of the total angular momentum is conserved:

$$\frac{dL^2}{dt} = 0 \quad -(46)$$

and in quantum mechanics becomes:

$$\hat{L}^2 \phi = l(l+1)\hbar^2 \phi \quad -(47)$$

where l is a quantum number.

From Eqs. (43) and (40):

$$L^2 = m^2 r^4 \dot{\theta}^2 + \frac{L_Z^2}{\sin^2\theta} \quad -(48)$$

so the three angular velocities of the system are:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(49)$$

$$\frac{d\phi}{dt} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(50)$$

$$\frac{d\theta}{dt} = \frac{1}{mr^2}\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2} \quad -(51)$$

It follows that:

$$\frac{d\beta}{d\phi} = \frac{L}{L_Z}\sin^2\theta \quad -(52)$$

and that:

$$\beta = \int \frac{L\, d\theta}{\left(L^2 - \frac{L_Z^2}{\sin^2\theta}\right)^{1/2}} = -\sin^{-1}\left(\frac{L\cos\theta}{(L^2 - L_Z^2)^{1/2}}\right) \quad -(53)$$

from computer algebra.

Similarly:

$$\beta = \int \frac{L}{L_Z}\sin^2\theta\, d\phi \quad -(54)$$

and using Eq. (53):

$$\cos\theta = -\frac{(L^2 - L_z^2)^{1/2}}{L} \sin\beta \quad -(55)$$

and:

$$\beta = \tan^{-1}\left(\frac{L}{L_z}\tan\phi\right) \quad -(56)$$

$$\phi = \tan^{-1}\left(\frac{L_z}{L}\tan\beta\right) \quad -(57)$$

Eqs. (53) and (56) give the sub orbits $\beta(\theta)$ and $\beta(\phi)$. Computer algebra was used to evaluate Eqs. (53) and (56). Finally the two angles of the spherical polar system are interrelated by:

$$\phi = -\frac{1}{2}\left(\sin^{-1}\left(\frac{(1+\cos\theta)L^2 - L_z^2}{|1+\cos\theta|(L^4 - L_z^2 L^2)}\right) + \sin^{-1}\left(\frac{(\cos\theta - 1)L^2 + L_z^2}{|(\cos\theta - 1)|(L^4 - L_z^2 L^2)}\right)\right) \quad -(58)$$

giving another sub orbit.

The main characteristics of some of these orbits are given in Section 3, together with an animation.

It is seen that basic concepts lead to wholly unexpected results and the emergence of the new subject area of three dimensional orbits constructed from universal gravitation in spherical polar coordinates.

The theory of orbits in spherical polar coordinates

M. W. Evans[*], H. Eckardt[†]
Civil List, A.I.A.S. and UPITEC

(www.webarchive.org.uk, www.aias.us,
www.atomicprecision.com, www.upitec.org)

3 Graphics and animation of some of the orbital features

In section 2 the equations for the calculation of orbits were derived. For a given angle θ, the corresponding angle ϕ is computable by Eq.(58), and the angle β depends on θ as derived in (53). With the latter equation, the three-dimensional elliptic orbit of beta,

$$r = \frac{\alpha}{1 + \epsilon \cos(\beta)}, \tag{59}$$

is defined. This is an orbital surface in 3D, graphed in Fig. 1. Only the half range of ϕ (0 to π) is used to visualize the three-dimensional structure in a clear way.

The true orbit is a line on this surface and is defined by the dependency $\beta(\theta, \phi)$. This is obtained in two steps. First $\phi(\theta)$ is determined from Eq.(58), then $\beta(\phi)$ is inserted into the elliptic equation (59), resulting in a functional dependence $r(\beta(\theta, \phi))$. The orbit extends from the upper to the lower end of the surface of Fig. 1 and is closed. It is graphed in Fig. 2. A projection of the orbit shows that it is planar. The edges appear because the 3D shape has non-differentiable points at the upper and lower end. (Notice that the internal cones in Fig. 1 do not belong to the surface but are artifacts of the graphics program). The orbit can be views in some animated versions on the website aias.us.

The orbit depends on the parameters L, L_Z, α and ϵ. So far we have used $L = 3$, $L_Z = 1$, $\alpha = 1$, $\epsilon = 0.5$. The transition $L \to L_Z$ is graphed in Fig. 3 in a projection to the XZ plane. It can be seen that the orbit shrinks to a 2D ellipse which is the two-dimensional limit for $\theta \to \pi/2$. The effect of the eccentricity ϵ has been demonstrated in Fig. 4. The rim of the orbital surface becomes sharper for increasing ϵ. If we set $\epsilon > 1$, the elipsoidal part changes

*email: emyrone@aol.com
†email: mail@horst-eckardt.de

into a hyperbolic surface as is to be expected, because the conic section equation (59) describes a hyperbola in that case.

In the last two images the sub-orbits have been studied. The dependencies $\phi(\theta)$, $\beta(\theta)$ and $\beta(\phi)$ are one-dimensional, therefore they are shown in a standard plot (Fig. 5). θ pertains from 0 to 2π which is the doubled standard range to give a closed curve. The sign of the curves had partially to be changed at $\theta = \pi$ to avoid jumps in the curves, this is an artifact of spherical polar coordinates. From Fig. 5 can be observed that the dependencies of the sub-orbits are nearly linear in the middle of their range. This means that the orbit is elliptic there. In Fig. 6 the corresponding sub-orbits $r(\theta), r(\phi), r(\beta)$ are graphed. They are elliptic in the middle of their ranges as predicted. At the borders there is no vertical tangent, showing that there is a non-differentiable point when this range is passed.

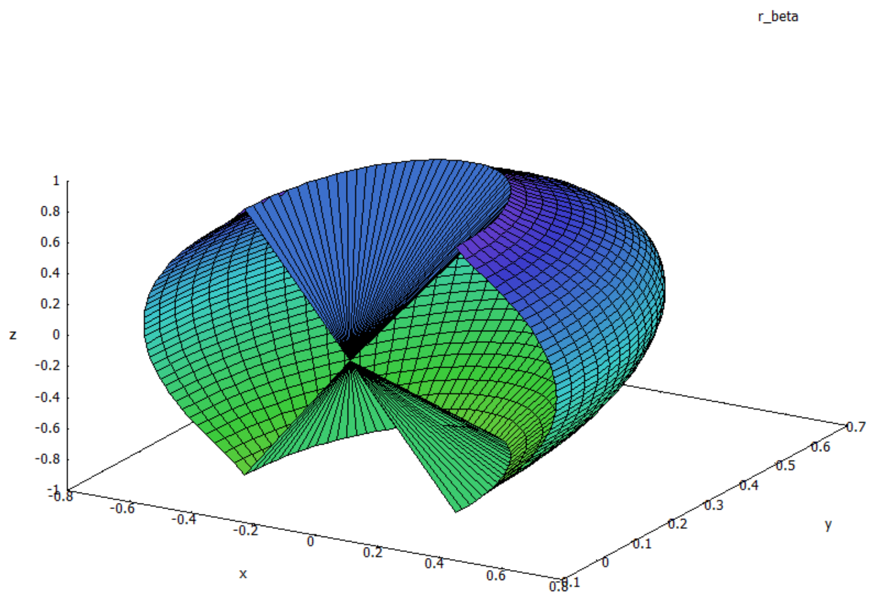

Figure 1: Orbital surface $r(\beta)$.

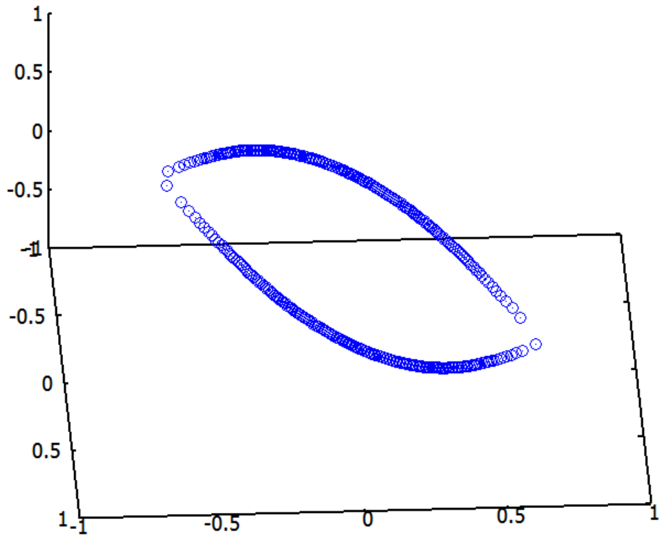

Figure 2: Orbit $r(\beta)$. This is a planar orbit over the surface of Fig. 1.

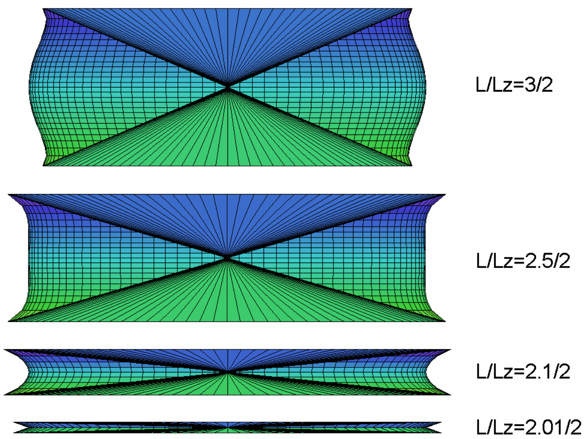

Figure 3: Orbital surface projections to the XZ plane for several ratios L/L_Z.

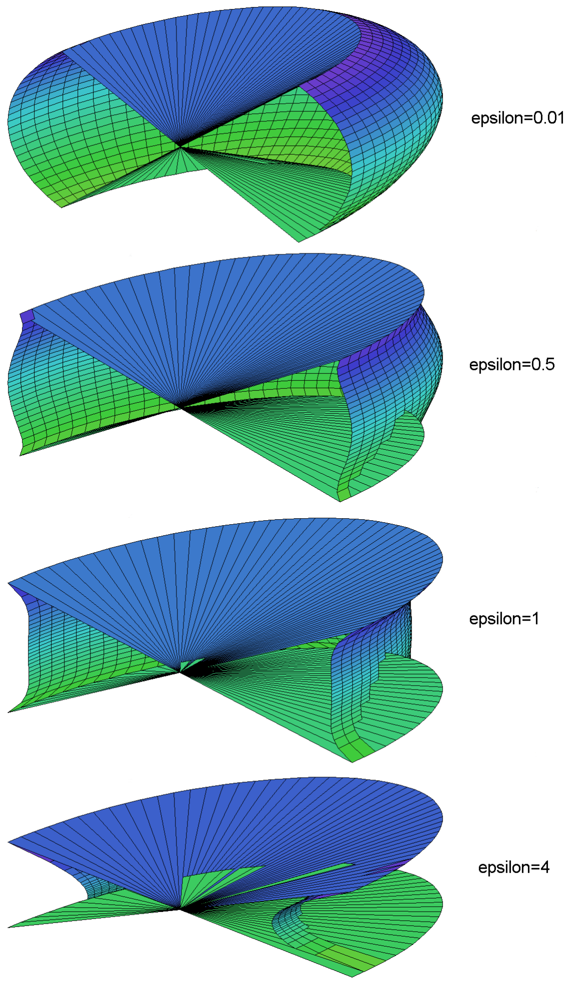

Figure 4: Orbital surfaces for several eccentricities ϵ.

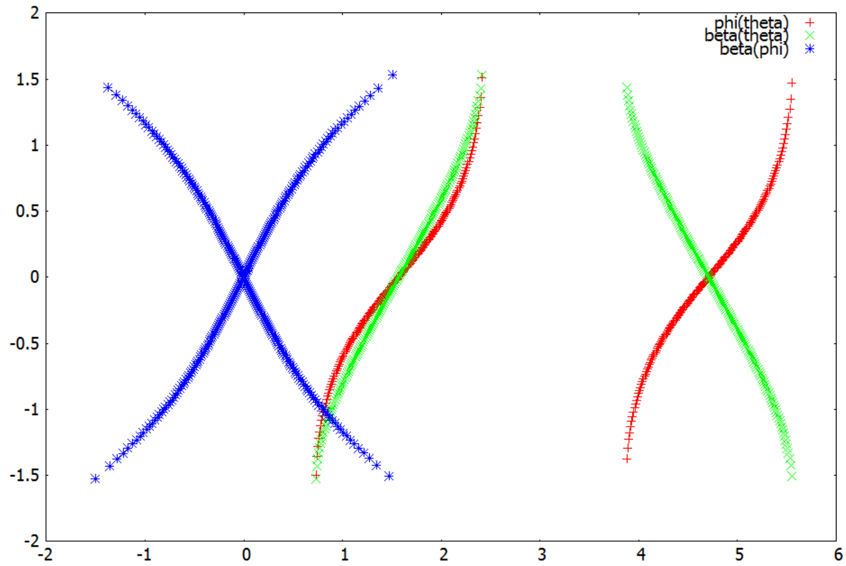

Figure 5: Sub-orbits $\phi(\theta)$, $\beta(\theta)$ and $\beta(\phi)$.

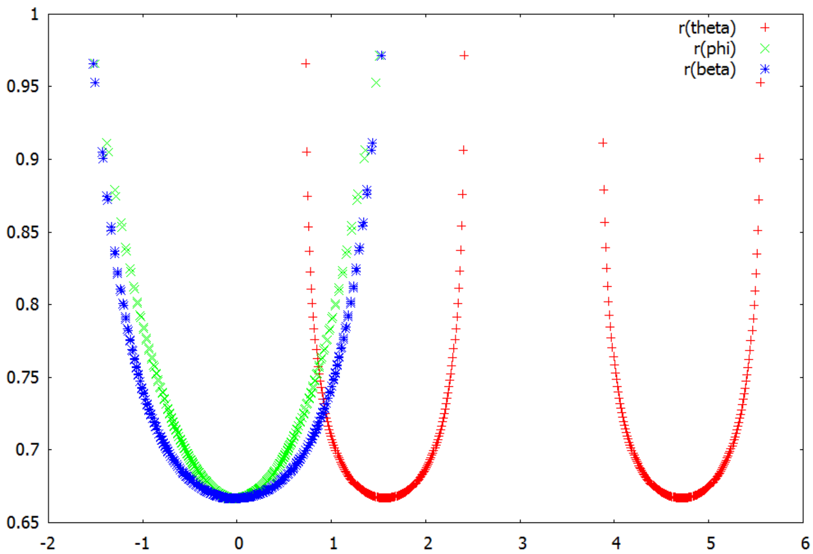

Figure 6: Radius functions $r(\theta)$, $r(\theta)$ and $r(\beta)$.

3. GRAPHICS AND ANIMATION OF SOME OF THE ORBITAL FEATURES

Section by Dr. Horst Eckardt

ACKNOWLEDGMENTS

The British Government is thanked for a Civil List Pension and the staff of AIAS and others for many interesting discussions. Dave Burleigh is thanked for posting, Alex Hill of translation and broadcasting, and Robert Cheshire for broadcasting.

REFERENCES

{1} M .W. Evans, Ed., "Definitive Refutations of the Einsteinian General Relativity" (Cambridge Scientific Science Publishing, CISP, 2012).

{2} M .W. Evans, H. Eckardt, D. W. Lindstrom and S. J. Crothers, "Principles of ECE Theory (open source on www.aias.us and in book format, in prep.)

{3} M .W. Evans, S. J. Crothers, H. Eckardt and K. Pendergast, "Criticisms of the Einstein Field Equation" (open source on www.aias.us. CISP, 2011).

{4} M .W. Evans, H. Eckardt and D. W. Lindstrom, "Generally Covariant Unified Field Theory" (Abramis Academic, 2005 to 2011 and open source on www.aias.us).

{5} L. Felker, "The Evans Equations of Unified Field Theory" (Abramis, 2007).

{6} M. W. Evans. H. Eckardt and D. W. Lindstrom, papers and plenary Serbian Academy of Sciences.

{7} M. W. Evans and L. B. Crowell, "Classical and Quantum Electrodynamics and the B(3) Field" (World Scientific, 2001).

{8} M .W. Evans and S. Kielich (Ed.), "Modern Nonlinear Optics" (Wiley Interscience, 1992, 1993, 1997 and 2001) in two volumes and six editions.

{9} M .W. Evans and J.-P. Vigier, "The Enigmatic Photon" (Kluwer, 1994 to 2002) in ten volumes hardback and softback.

{10} M .W. Evans and A. A. Hasanein, "The Photomagneton in Quantum Field Theory" (World Scientific, 1994).

{11} E. G. Milewski (Ed.), "The Vector Analysis Problem Solver" (Res. Ed. Association, New York, 1987).

{12} J. B. Marion and S. T. Thornton, "Classical Dynamics of Particles and Systems" (Harcourt, New York, 188, 3rd. Edition)