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A Generally Covariant Wave Equation For Grand Unified Field Theory

Summary. A generally covariant wave equation is derived geometrically for grand unified field theory. The equation states most generally that the covariant d'Alembertian acting on the vielbein vanishes for the four fields which are thought to exist in nature: gravitation, electromagnetism, weak field and strong field. The various known field equations are derived from the wave equation when the vielbein is the eigenfunction. When the wave equation is applied to gravitation the wave equation is the eigenequation of wave mechanics corresponding to Einstein's field equation in classical mechanics, the vielbein eigenfunction playing the role of the quantized gravitational field. The three Newton laws, Newton's law of universal gravitation, and the Poisson equation are recovered in the classical and nonrelativistic, weak-field limits of the quantized gravitational field. The single particle wave-equation and Klein-Gordon equations are recovered in the relativistic, weak-field limit of the wave equation when scalar components are considered of the vielbein eigenfunction of the quantized gravitational field. The Schrödinger equation is recovered in the non-relativistic, weak-field limit of the Klein-Gordon equation). The Dirac equation is recovered in this weak-field limit of the quantized gravitational field (the non-relativistic limit of the relativistic, quantized gravitational field when the vielbein plays the role of the spinor. The wave and field equations of $O(3)$ electrodynamics are recovered when the vielbein becomes the relativistic dreibein (triad) eigenfunction whose three orthonormal space indices become identified with the three complex circular indices (1), (2), (3), and whose four spacetime indices are the indices of non-Euclidean spacetime (the base manifold). This dreibein is the potential dreibein of the $O(3)$ electromagnetic field (an electromagnetic potential four-vector for each index (1), (2), and (3)). The wave equation of the parity violating weak field is recovered when the orthonormal space indices of the relativistic dreibein eigenfunction are identified with the indices of the three massive weak field bosons. The wave equation of the strong field is recovered when the orthonormal space indices of the relativistic vielbein eigenfunction become the eight indices defined by the group generators of the $SU(3)$ group.

Key words: generally covariant equation, grand unified field theory, gravitation, higher symmetry electromagnetism, $O(3)$ electrodynamics, weak field, strong field.

4.1 Introduction

Recently [1] a generally covariant classical field equation has been proposed for the unification of the classical gravitational and electromagnetic fields by considering the metric four-vector q_μ in non-Euclidean spacetime. In this Letter the corresponding equation in wave (or quantum) mechanics is derived by considering the action of the covariant d'Alembertian operator on the metric four-vector considered as the eigenfunction. By deriving a metric compatibility equation for the metric four-vector a wave equation is obtained from a fundamental geometrical property: the covariant d'Alembertian acting on the metric vector vanishes in non-Euclidean spacetime. This geometrical result is also true when the eigenfunction is a symmetric or anti-symmetric metric tensor [1], and, most generally, when the eigenfunction is a vielbein [2]. The wave equation with symmetric metric tensor as eigenfunction is the direct result of the latter's metric compatibility equation, and the wave equation with vielbein as eigenfunction is the result of the tetrad postulate [2]. The latter is a fundamental result of geometry irrespective of metric compatibility and whether or not the metric tensor is torsion free. The wave equation can therefore be constructed as an eigenequation from geometry with different types of eigenfunction. This is achieved in this Letter by expressing the covariant d'Alembertian operator as a sum of the flat space d'Alembertian operator $\Box$ plus a term dependent on the non-Euclidean nature of spacetime. The latter term is shown to be a scalar curvature R which is identified as the eigenvalue. The eigenoperator is therefore the operator $\Box$, and the wave equation is a fundamental geometrical property of non-Euclidean spacetime [1,2]. Most generally the eigenfunction is the vielbein [2] $e^a{}_\mu$, which relates an orthonormal basis (Latin index) to a coordinate basis (Greek index), and the generally covariant wave equation is the eigenequation

$$(\Box + kT)e^a{}_\mu = 0. \tag{4.1}$$

The Einstein field equation [1-3] of gravitational general relativity can be written in the contracted form [1-4]

$$R = -kT, \tag{4.2}$$

where R and T are obtained [4] from the curvature tensor and the canonical energy momentum tensor by index contraction. If $q_{\mu\nu}{}^{(S)}$ denotes the symmetric metric tensor defined [1] by

$$q_{\mu\nu}{}^{(S)} = q_\mu q_\nu \tag{4.3}$$

then

$$R = q^{\mu\nu(S)} R_{\mu\nu}, \quad T = q^{\mu\nu(S)} T_{\mu\nu}, \tag{4.4}$$

where $R_{\mu\nu}$ and $T_{\mu\nu}$ are also symmetric tensors. Therefore the generally covariant wave equation is

$$(\Box + kT)e^a{}_\mu = 0, \quad kT = -R. \tag{4.5}$$

It can be seen that Eq. (4.5) has the form of the well known second-order wave equations of dynamics and electrodynamics, such as the single particle wave equation, the Klein-Gordon, Dirac, Proca, and d'Alembert [5] and non-relativistic limiting forms, such as the Schrödinger equation, and, in the classical limit, the Poisson and Newton equations. The use of the vielbein as eigenfunction has several well known advantages [2]:

(4.1) The tetrad postulate:

$$D_\nu e^a{}_\mu = 0, \tag{4.6}$$

where D_ν denotes the covariant derivative [2], is true for any connection, whether or not it is metric compatible or torsion free.
(4.2) The use of the vielbein as eigenfunction allows spinors to be analyzed in non-Euclidean spacetime, and this is essential to derive the Dirac equation from Eq. (4.5).
(4.3) The index a of the vielbein can be identified with the internal index of gauge theory [2,5], and this property is essential if Eq. (4.5) is to be an equation of grand unified field theory.
(4.4) Vielbein theory is highly developed and is closely related to Cartan-Maurer theory, a generalization of Riemann geometry [2].

The structure group of the tangent bundle in the four dimensional spacetime base manifold is $GL(4, R)$ [2], the group of real invertible 4×4 matrices. In a Lorentzian metric this reduces to the Lorentz group $SO(3, 1)$. The fibers of the fiber bundle are tied together with ordinary rotations [2] and the structure group of the new bundle is $SO(3)$, the group of rotations in three space dimensions without the assumption of parity symmetry. The electromagnetic potential is defined on this bundle by the dreibein $A^a{}_\mu$, where a is (1), (2) or (3), the indices of the complex circular representation of three dimensional space. The evolution of electrodynamics in this way, as a gauge theory with $O(3)$ gauge group symmetry, where $O(3)$ is the group of rotations in three dimensions with parity symmetry, began with the proposal of the $\boldsymbol{B}^{(3)}$ field [6] as being responsible for the inverse Faraday effect in all materials (phase free magnetization by circularly polarized electromagnetic radiation). Maxwell Heaviside electrodynamics is a gauge field theory with no internal indices, and whose internal gauge group symmetry is $U(1)$ [7-12]. After a decade of development it is known [12] that there are numerous instances in which $O(3)$ electrodynamics surpasses $U(1)$ electrodynamics in its ability to describe experimental data, for example data from interferometry, reflection, physical optics in general, the inverse Faraday effect, and its resonance equivalent, radiatively induced fermion resonance [7-12]. Therefore many data are now known which indicate that the electromagnetic sector of grand unified field theory is described by an $O(3)$ symmetry gauge field theory, not $U(1)$. In the development of $O(3)$ electrodynamics the connection on the internal fiber bundle of gauge theory was identified for the first time as the connection on the tangent bundle of general relativity [2]. This is an essential step towards the evolution of a simple and powerful unified field theory as embodied in Eq. (4.5) of this Letter. The tangent bundle is defined with respect to the base manifold, which is four dimensional non-Euclidean spacetime [1]. Prior to the development of higher symmetry electrodynamics, and generally covariant electrodynamics [1,13,14] the tangent bundle of general relativity [2] was not identified with the fiber bundle of gauge theory, in other words the internal index of gauge theory was thought to be the index of an abstract space unrelated to spacetime [2]. In $O(3)$ electrodynamics the internal index $a = (1), (2), (3)$ represents a physical orthonormal space tangential to the base manifold and in the basis ((1),(2),(3)) it is possible to define unit vectors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}$ which define a tangent space, a space that is orthonormal to the base manifold (non-Euclidean, four dimensional spacetime). It therefore becomes possible to invoke the dreibein, or triad, as described already, with the three Latin indices (a) representing the orthogonal space and the four Greek indices (μ) the base manifold. The indices $a = (1), (2), (3)$ can be used to define the unit vector system in curvilinear coordinate analysis [1,14,15]. One of the unit vectors, e.g., $\boldsymbol{e}^{(1)}$, is a unit tangent vector to the curve, and the other two, $\boldsymbol{e}^{(2)}$ and $\boldsymbol{e}^{(3)}$, are mutually orthogonal to $\boldsymbol{e}^{(1)}$. This procedure cures two fundamental inconsistencies of field theory as it stands at present:

(1) The gravitational field is non-Euclidean space time in general relativity, while the other three fields (electromagnetic, weak and strong) are entities superimposed on flat or Euclidean spacetime.
(2) The $U(1)$ electromagnetic field has an Abelian and linear character, while the other three fields are non-Abelian and nonlinear[2].

Therefore the internal gauge space of $O(3)$ electrodynamics is identified with a tangent space in the complex circular basis ((1),(2),(3)), a basis chosen to represent circular polarization, a well known empirical property of electromagnetic radiation [5-14]. This allows electromagnetism to be developed as a theory of general relativity [1], using $O(3)$ symmetry covariant derivatives, which become spin affine connections in vielbein theory [2]. The $O(3)$ electromagnetic field tensor becomes a Cartan Maurer torsion tensor [2] which is defined with a spin affine connection on the tangent bundle of general relativity. These properties are contained with Eq. (4.5), together with the ability to describe the gravitational, weak and strong fields. Therefore, Eq. (4.5) is a generally covariant wave equation of grand unified field theory.

In Sec. 2 the wave equation (4.5) is derived for various forms of the eigenfunction using metric compatibility equations for the metric vector q_μ and the symmetric and anti-symmetric metric tensors $q_\mu q_\nu$ and $q_\mu \wedge q_\nu$ [1] and using the tetrad postulate [2] for the vielbein $e^a{}_\mu$. In Sec. 3 the equation of parallel transport and the geodesic equation [2-4] are written in terms of the metric four-vector q_μ, and the equation of metric compatibility of the metric four vector shown to be a solution of the geodesic equation. In Sec. 4

the Poisson equation and the Newtonian equations are derived in the weak field limit of gravitational theory. In Sec. 5 the second order wave equations are derived from Eq. (4.5) in various limits for the four known fields of nature. Finally, Sec. 6 is a discussion of some of the many possible avenues for further work based on Eq. (4.5) and its classical equivalent given in Ref. [1].

4.2 Derivation Of The Generally Covariant Wave Equation

The wave equation is based on the following expression for the covariant d'Alembertian operator:

$$D^\rho D_\rho = \Box + D^\mu \Gamma^\rho_{\ \mu\rho}\,, \tag{4.7}$$

where

$$D^\mu \Gamma^\rho_{\ \mu\rho} = \partial^\mu \Gamma^\rho_{\ \mu\rho} + \Gamma^{\mu\rho}_{\ \ \lambda}\,\Gamma^\lambda_{\ \mu\rho} \tag{4.8}$$

is the covariant derivative of the index contracted Christoffel symbol $\Gamma^\rho_{\ \mu\rho}$. Equation (4.7) is derived by first considering the commutator [2] $[D_\mu, D_\nu]$ acting on the inverse metric vector q^ρ [1]:

$$\begin{aligned}
&[D_\mu, D_\nu]q^\rho = D_\mu D_\nu q^\rho - D_\nu D_\mu q^\rho \\
&= \partial_\mu(D_\nu q^\rho) - \Gamma^\lambda_{\ \mu\nu}\,D_\lambda q^\rho + \Gamma^\rho_{\ \mu\sigma}\,D_\nu q^\sigma - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu q^\rho + (\partial_\mu \Gamma^\rho_{\ \nu\sigma})q^\sigma + \Gamma^\rho_{\ \nu\sigma}\,\partial_\mu q^\sigma - \Gamma^\lambda_{\ \mu\nu}\,\partial_\lambda q^\rho \\
&- \Gamma^\lambda_{\ \mu\nu}\,\Gamma^\rho_{\ \lambda\sigma}\,q^\sigma + \Gamma^\rho_{\ \mu\sigma}\,\partial_\nu q^\sigma + \Gamma^\rho_{\ \mu\sigma}\,\Gamma^\sigma_{\ \nu\lambda}\,q^\lambda - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu \Gamma^\rho_{\ \nu\sigma} - \partial_\nu \Gamma^\rho_{\ \mu\sigma} + \Gamma^\rho_{\ \mu\lambda}\,\Gamma^\lambda_{\ \nu\sigma} - \Gamma^\rho_{\ \nu\lambda}\,\Gamma^\lambda_{\ \mu\sigma})q^\sigma \\
&- 2(\Gamma^\lambda_{\ \mu\nu} - \Gamma^\lambda_{\ \nu\mu})D_\lambda q^\rho \\
&= R^\rho_{\ \sigma\mu\nu} q^\sigma - T^\lambda_{\ \mu\nu}\,D_\lambda q^\rho,
\end{aligned} \tag{4.9}$$

where $R^\rho_{\ \sigma\mu\nu}$ is the Riemann tensor and $T^\lambda_{\ \mu\nu}$ is the torsion tensor. Consideration of the symmetry of Eq. (4.9), and contracting indices, $\rho = \sigma$, leads to the result

$$D^\mu D_\mu = \partial^\mu \partial_\mu + \partial^\mu \Gamma^\rho_{\ \mu\rho} + \Gamma^{\mu\rho}_{\ \ \lambda}\,\Gamma^\lambda_{\ \mu\rho} + 2\Gamma^\lambda_{\ \mu\mu}\,D_\lambda. \tag{4.10}$$

In this expression the Christoffel symbols are defined as [2]

$$(\Gamma_\mu)^\rho_{\ \rho} := \Gamma^\rho_{\ \mu\rho}\,, \quad (\Gamma_\mu)^\rho_{\ \lambda} := \Gamma^\rho_{\ \mu\lambda}\,, \quad \text{etc.}, \tag{4.11}$$

but, by convention [2], the brackets are omitted in the notation. We follow this convention in the rest of this paper. For any vector V^ν,

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\ \mu\lambda}\,V^\lambda. \tag{4.12}$$

We therefore can write

$$D^{\mu}\Gamma^{\rho}_{\ \mu\rho} = \partial^{\mu}\Gamma^{\rho}_{\ \mu\rho} + \Gamma^{\mu\rho}_{\ \ \lambda}\Gamma^{\lambda}_{\ \mu\rho}. \tag{4.13}$$

The covariant d'Alembertian operator is therefore in general

$$D^{\mu}D_{\mu} = \Box + D^{\mu}\Gamma^{\rho}_{\ \mu\rho} + 2\Gamma^{\lambda}_{\ \mu\mu}D_{\lambda} \tag{4.14}$$

and can be thought of qualitatively as "half the Riemann tensor plus half the torsion tensor." This is a geometrical result independent of any considerations of field theory.

Equation (4.5), the wave equation for the vielbein as eigenfunction, follows from the tetrad postulate [2]:

$$D_{\rho}e^{a}_{\ \mu} = 0, \tag{4.15}$$

which holds whether or not the connection is metric-compatible or torsion-free. Differentiating Eq. (4.15) covariantly gives Eq. (4.5):

$$\begin{aligned} D^{\rho}(D_{\rho}e^{a}_{\ \mu}) &:= D^{\rho}D_{\rho}e^{a}_{\ \mu} = \left(\Box + D^{\mu}\Gamma^{\rho}_{\ \mu\rho} + 2\Gamma^{\lambda}_{\ \mu\mu}D_{\lambda}\right)e^{a}_{\ \mu} \\ &= \left(\Box + D^{\mu}\Gamma^{\rho}_{\ \mu\rho}\right)e^{a}_{\ \mu} = 0. \end{aligned} \tag{4.16}$$

Equation (4.16) is therefore a geometrical result which is independent of any assumptions made concerning the Christoffel symbol and its relation to the metric tensor [2] or metric vector [1]. The covariant d'Alembertian operator appearing in the wave equation (4.16) is the sum of the flat spacetime d'Alembertian $\Box$ and the term $D^{\mu}\Gamma^{\rho}_{\ \mu\rho}$. The latter is identified as *scalar curvature* (R) because it has the units of inverse square meters and is defined by an index contraction [2]. The scalar curvature R is obtained conventionally by contracting indices in the Riemann tensor. Carroll [2], for example, defines R as follows:

$$R := q^{\sigma\nu(S)}R^{\lambda}_{\ \sigma\lambda\nu}, \tag{4.17}$$

where the Ricci tensor is [2]

$$R_{\mu\nu} := R^{\lambda}_{\ \mu\lambda\nu} \tag{4.18}$$

and the Riemann tensor with lowered indices is [2]

$$R_{\rho\sigma\mu\nu} := q_{\rho\lambda}^{\ \ (S)}R^{\lambda}_{\ \sigma\mu\nu}. \tag{4.19}$$

However, Sachs [16] gives a different definition of the Ricci tensor:

$$R_{\kappa\rho} := q^{\mu\lambda(S)}R_{\mu\kappa\rho\lambda}; \tag{4.20}$$

so, assuming Eq. (4.19) and contracting indices $\alpha = \lambda$:

$$R_{\kappa\rho} = \delta^{\lambda}_{\lambda}R^{\lambda}_{\ \kappa\rho\lambda} = q^{\mu\lambda(S)}q_{\mu\alpha}^{\ \ (S)}R^{\alpha}_{\ \kappa\rho\lambda} \tag{4.21}$$

Comparing Eqs. (4.18) and (4.21), and it is seen that the definition of the scalar curvature R is a matter of convention, and is not standardized. Different authors give different definitions. Therefore the R that appears in the Einstein equation with contracted indices, Eq. (4.2), is a matter of convention. Furthermore, the minus sign that appears in Eq. (4.2) is also a matter of convention, Einstein himself [4] used the equation in the form $R = kT$, without a minus sign. In the rest of this paper we will use the contemporary [2] convention $R = -kT$. The general rule is that the scalar curvature R is found by a contraction of indices in the Riemann tensor, which has several well-known symmetry properties [2], for example–it is anti-symmetric in its last two indices. Using the following choice of index contraction:

$$R := R^{\mu}{}_{\nu\mu\nu} = \partial_{\mu}\Gamma^{\mu}{}_{\nu\nu} - \partial_{\nu}\Gamma^{\mu}{}_{\mu\nu} + \Gamma^{\mu}{}_{\mu\nu}\Gamma^{\nu}{}_{\nu\nu} - \Gamma^{\mu}{}_{\nu\nu}\Gamma^{\nu}{}_{\mu\nu}\,, \tag{4.22}$$

it can be seen that in this convention the scalar curvature is

$$\begin{aligned} R &:= \partial_{\mu}\Gamma^{\mu}{}_{\nu\nu} + \Gamma^{\mu}{}_{\mu\nu}\Gamma^{\nu}{}_{\nu\nu} - \left(\partial_{\nu}\Gamma^{\mu}{}_{\mu\nu} + \Gamma^{\mu}{}_{\nu\nu}\Gamma^{\nu}{}_{\mu\nu}\right) \\ &= D_{\mu}\Gamma^{\mu}{}_{\nu\nu} - D_{\nu}\Gamma^{\mu}{}_{\mu\nu}\,. \end{aligned} \tag{4.23}$$

Comparing Eqs. (4.16) and (4.23), it is deduced that the scalar curvature

$$R := -D^{\mu}\Gamma^{\rho}{}_{\mu\rho} \tag{4.24}$$

that appears in the definition of the covariant d'Alembertian is, qualitatively, "half" of the scalar curvature in Eq. (4.3), obtained directly from the Riemann tensor. This result is consistent with the fact that the covariant d'Alembertian is, qualitatively (or roughly speaking), half the Riemann plus torsion tensors. If the Christoffel symbol is assumed to be symmetric in its lower two indices (as in the convention in standard general relativity [2]) then the scalar curvature R defined in Eq. (4.24) becomes the second term in Eq. (4.23). If the Christoffel symbol is anti-symmetric in its lower two indices, as in the definition of the torsion tensor (Eq. (4.9)), then the second term in Eq. (4.23) is the negative of the definition appearing in Eq. (4.24). Importantly, however, Eq. (4.16) is valid whatever the symmetry of the Christoffel symbol, because Eq. (4.16) is the direct result of the tetrad postulate, Eq. (4.6). Therefore Eq. (4.16) is true for curved spacetime (gravitation) and twisted or torqued spacetime (electromagnetism). Using Eq. (4.2), we deduce the wave equation in the form

$$(\Box + kT)e^{a}{}_{\mu} = 0, \tag{4.25}$$

where

$$D^{\mu}\Gamma^{\rho}{}_{\mu\rho} = kT = -R. \tag{4.26}$$

A less generally valid wave equation can be obtained with the symmetric metric tensor of the Einstein field equation [1-4] as eigenfunction. This wave equation follows from the metric compatibility condition [2]:

$$D_\rho q_{\mu\nu}{}^{(S)} = 0. \tag{4.27}$$

Differentiating Eqs. (4.17) covariantly leads to the wave equation as the eigen equation:

$$D^\rho D_\rho q_{\mu\nu}{}^{(S)} = (\Box + kT) q_{\mu\nu}{}^{(S)} = 0, \tag{4.28}$$

where $q_{\mu\nu}{}^{(S)}$ is the eigenfunction. A third type of wave equation can be obtained using the definition [1]

$$q_{\mu\nu}{}^{(S)} = q_\mu q_\nu. \tag{4.29}$$

Covariant differentiation of products is defined by the Leibniz theorem [2]; therefore the metric compatibility of the symmetric metric tensor, Eq. (4.27), implies that

$$D_\rho(q_\mu q_\nu) = q_\mu(D_\rho q_\nu) + (D_\rho q_\mu)q_\nu = 0, \tag{4.30}$$

for the first derivative, and

$$\begin{aligned} D^2(q_\mu q_\nu) &:= (D^\rho D_\rho)(q_\mu q_\nu) \\ &= q_\mu D^2 q_\nu + 2(D_\rho q_\mu)(D_\rho q_\nu) + q_\nu D^2 q_\mu \\ &= 0 \end{aligned} \tag{4.31}$$

for the second derivative. A self consistent solution of Eqs. (4.30) and (4.31) is

$$D_\mu q_\nu = 0, \tag{4.32}$$

which is a metric compatibility condition for the metric vector q_μ. Differentiating Eq. (4.32) covariantly gives the wave equation as an eigenequation with the metric vector q_μ as eigenfunction:

$$D^\rho D_\rho q_\mu = (\Box + kT) q_\mu = 0. \tag{4.33}$$

Finally, it may be shown similarly that there exists a wave equation with the anti-symmetric metric $q_{\mu\nu}{}^{(A)} = q_\mu \wedge q_\nu$ as eigenfunction, i.e.:

$$(\Box + kT) q_{\mu\nu}{}^{(A)} = 0. \tag{4.34}$$

4.3 Fundamental Equations In Terms Of The Metric Vector

The equation of metric compatibility (4.32) can be derived independently as a solution of the equation of parallel transport [2] written for the inverse metric four-vector q^μ:

$$\frac{Dq^\mu}{ds} := \frac{dq^\mu}{ds} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\nu}{ds} q^\lambda = 0, \tag{4.35}$$

where dx^ν/ds is the tangent vector to q^μ. Here

$$(ds)^2 = q^\mu q^\nu dx_\mu dx_\nu \tag{4.36}$$

is the square of the line element in curvilinear coordinates [1]. The geodesic equation for q^μ is

$$\frac{D}{ds}\left(\frac{dq^\mu}{ds}\right) = 0. \tag{4.37}$$

Now use the chain rule [17] if $u = f(x, y)$; then

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}; \tag{4.38}$$

so, if $q^\mu = q^\mu(x^\nu)$, then

$$\frac{dq^\mu}{ds} = \frac{\partial q^\mu}{\partial x^\nu}\frac{dx^\nu}{ds}. \tag{4.39}$$

Using Eq. (4.39) in Eq. (4.35), one gets

$$\left(\frac{\partial q^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\lambda}\, q^\lambda\right)\frac{dx^\nu}{ds} = 0, \tag{4.40}$$

i.e.,

$$(D_\nu q^\mu)\frac{dx^\nu}{ds} = 0. \tag{4.41}$$

In general $dx^\nu/ds \neq 0$, so Eq. (4.32), the metric compatibility condition for q_μ, is a solution of Eq. (4.41). Therefore the equation of metric compatibility for q_μ can be derived as a solution of the equation of parallel transport for q^μ. The geodesic equation (4.37) follows from the equation of parallel transport (4.35), so the metric compatibility equation for q_μ is a special case of the geodesic equation for q^μ. Using Eq. (4.39), the geodesic equation becomes

$$\left(\frac{D}{ds}\left(\frac{\partial q^\mu}{\partial x^\nu}\right)\right)\frac{\partial x^\nu}{\partial s} + \frac{\partial q^\mu}{\partial x^\nu}\left(\frac{D}{ds}\left(\frac{\partial x^\nu}{\partial s}\right)\right) = 0. \tag{4.42}$$

But the geodesic equation can be written for any vector V^μ, and so

$$\frac{D}{ds}\left(\frac{dx^\nu}{ds}\right) = 0. \tag{4.43}$$

It therefore follows that

$$\frac{D}{ds}\left(\frac{\partial q^\mu}{\partial x^\nu}\right) = 0. \tag{4.44}$$

As shown in Ref. [1], the gravitation and electromagnetism can be described from a novel generally covariant field equation for q_μ:

$$R_\mu - \frac{1}{2}Rq_\mu = kT_\mu. \tag{4.45}$$

4.4 Derivation Of The Poisson And Newton Equations

The Poisson equation of gravitation can be derived straightforwardly in the weak field limit [1-4] from the wave equation for an eigenfunction, for example Eq. (4.33) which can be written as the two equations:

$$(\Box + kT)q_o = 0, \tag{4.46}$$

$$(\Box + kT)q_i = 0, \quad i = 1, 2, 3. \tag{4.47}$$

Using

$$\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{4.48}$$

Eq. (4.46) becomes

$$\nabla^2 q_o = kTq_o \tag{4.49}$$

for a quasi-static q_o. In the weak-field limit [1-4]:

$$q_o = \epsilon + \eta_o = 1 + \eta_o, \quad \eta_o \ll 1, \tag{4.50}$$

where ϵ_μ is the unit four-vector. Therefore Eq. (4.49) becomes

$$\nabla^2 \eta_o = kTq_o \sim kT. \tag{4.51}$$

This is the Poisson equation

$$\nabla^2 \Phi = 4\pi G\rho \tag{4.52}$$

if

$$\Phi = \frac{1}{2}c^2\eta_o \tag{4.53}$$

is the gravitational potential, if

$$\rho = T = m/V \tag{4.54}$$

is the rest energy density, and if G is Newton's gravitational constant, related to Einstein's constant by:

$$k = 8\pi G/c^2. \tag{4.55}$$

Newton's law, his theory of universal gravitation, and the equivalence of inertial and gravitational mass are all contained within the metric compatibility condition

$$\frac{\partial q^\mu}{\partial x^\nu} = -\Gamma^\mu{}_{\nu\lambda}\, q^\lambda. \tag{4.56}$$

Multiplying Eq. (4.56) on both sides by q_λ and using

$$\begin{aligned} \left(\Gamma^\mu{}_{\nu 0}\, q^0\right) q_0 &= \Gamma^\mu{}_{\nu 0}\,(q^0 q_0) = \Gamma^\mu{}_{\nu 0}\,, \\ \left(\Gamma^\mu{}_{\nu 1}\, q^1\right) q_1 &= \Gamma^\mu{}_{\nu 1}\,(q^1 q_1) = -\Gamma^\mu{}_{\nu 1}\,, \text{ etc.,} \end{aligned} \tag{4.57}$$

a unique equation is obtained for the Christoffel symbol in terms of the metric vector, irrespective or whether or not the metric vector is torsion-free:

$$\Gamma^{\mu}{}_{\nu o} = -q_o \partial_\nu q^\mu, \quad \Gamma^{\mu}{}_{\nu i} = q_i \partial_\nu q^\mu, \quad i = 1, 2, 3. \tag{4.58}$$

(The well-known equation relating the Christoffel symbol to the symmetric metric tensor is more intricate and less useful, because it is derived on the assumption of a torsion-free metric, i.e., that the Christoffel symbol is symmetric in its lower two indices. It is:

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} q^{\sigma\rho(S)} \left(\partial_\mu q_{\nu\rho}{}^{(S)} + \partial_\nu q_{\rho\mu}{}^{(S)} - \partial_\rho q_{\mu\nu}{}^{(S)} \right), \tag{4.59}$$

where $q^{\sigma\rho(S)}$ is the inverse of the symmetric metric tensor. The metric tensors are defined by [2]

$$q^{\mu\nu(S)} q_{\nu\sigma}{}^{(S)} = \delta^\mu_\sigma, \tag{4.60}$$

where

$$\delta^\mu_\sigma = \begin{cases} 1, & \mu = \sigma, \\ 0, & \mu \neq \sigma, \end{cases} \tag{4.61}$$

is the Kronecker delta. In non-Euclidean spacetime, the elements of $q^{\mu\nu(S)}$ and $q_{\mu\nu}{}^{(S)}$ are not the same in general [2].)

In the Newtonian limit, the particle velocities are much smaller than c, so [2-4]:

$$dx^i/d\tau \ll dt/d\tau \sim 1. \tag{4.62}$$

Using the chain rule for the left-hand side of Eq. (4.56), with proper time τ as an affine parameter [2], we obtain

$$\frac{\partial q^\mu}{\partial x^\nu} = \frac{d\tau}{dx^\nu} \frac{dq^\mu}{d\tau} \to \frac{1}{c} \frac{d\tau}{dt} \frac{dq^\mu}{d\tau} = \frac{1}{c} \frac{dq^\mu}{dt}. \tag{4.63}$$

Consider now the identity obtained from the equation of metric compatibility, Eq. (4.56):

$$\frac{\partial q^\mu}{\partial x^\nu} := \frac{\partial q^\mu}{\partial x^\nu}. \tag{4.64}$$

Using the chain rule in the weak-field limit, the left-hand side of Eq. (4.64) becomes

$$\frac{\partial q^\mu}{\partial x^\nu} \to \frac{1}{c} \frac{\partial q^\mu}{\partial t}. \tag{4.65}$$

If we consider the four-vector defined by

$$x^\mu = (x^0, x^1, x^2, x^3), \tag{4.66}$$

then the metric vector is defined by ($q^0 := q^\mu(\mu = 0)$, etc.)

$$q^0 = \frac{\partial x^\mu}{\partial x^0}, \quad q^1 = \frac{\partial x^\mu}{\partial x^1}, \quad q^2 = \frac{\partial x^\mu}{\partial x^2}, \quad q^3 = \frac{\partial x^\mu}{\partial x^3}; \tag{4.67}$$

therefore the left-hand side of Eq. (4.64) becomes for $\mu = 0$:

$$\frac{1}{c}\frac{\partial q^0}{\partial t} = \frac{1}{c^2}\frac{\partial^2 x^\nu}{\partial t^2} \tag{4.68}$$

in the weak-field or Newtonian limit. In this limit the metric can be considered as a perturbation of the flat spacetime metric [2]:

$$q^0 = \left(1 - \frac{1}{2}\eta^0\right) \sim 1. \tag{4.69}$$

The gravitational field in the Newtonian limit is quasi-static, and the position vector is dominated by its time-like component, so

$$\frac{\partial q^0}{\partial x^\nu} \to -\frac{1}{2}\frac{\partial \eta^0}{\partial x^\nu}. \tag{4.70}$$

Equating left-hand and right-hand sides of the identity (4.64), gives, in the Newtonian approximation,

$$\frac{d^2 x^i}{dt^2} = -\frac{c^2}{2}\frac{\partial \eta^0}{\partial x^i}, \tag{4.71}$$

which is Newton's second law combined with the Newtonian theory of universal gravitation. It is seen that the equivalence of gravitational and inertial mass implied by Eq. (4.71) is a consequence of geometrical identity (4.64). This is a powerful and original result, obtained from the novel equation of metric compatibility (4.56).

Using the definition (4.53) for the Newtonian potential Φ, Eq. (4.71) can be written in the familiar form

$$\frac{d^2 \boldsymbol{r}}{dt^2} = -\boldsymbol{\nabla}\Phi, \tag{4.72}$$

which is equivalent to the inverse square law of Newton

$$\boldsymbol{F} = m\frac{d^2 \boldsymbol{r}}{dt^2} = -G\frac{mM}{r^2}\boldsymbol{k}. \tag{4.73}$$

From Eq. (4.56) with $\mu = 0$, it can be seen that Eq. (4.72) can be expressed as one Christoffel symbol:

$$\frac{\partial q^0}{\partial x^\nu} = -\Gamma^0{}_{\nu 0}\, q^0 \sim -\Gamma^0{}_{\nu 0}\,, \tag{4.74}$$

an equation which shows that the equivalence of gravitational and inertial mass is a geometrical result, the equation of metric compatibility, (4.56), which also leads to the generally covariant wave equation (4.33), and self-consistently, to the Poisson Eq. (4.52) in the Newtonian approximation. If we write Eq. (4.28) as

$$(\Box + kT)g_{\mu\nu} = 0, \tag{4.75}$$

i.e., using the standard notation $g_{\mu\nu} = g_{\mu\nu}{}^{(S)}$ for the symmetric metric, the weak-field or Newtonian approximation gives

$$(\Box + km/V)\, g_{oo} = 0, \tag{4.76}$$

where $T = m/V$. If g_{oo} is considered to be quasistatic, Eq. (4.76) reduces to

$$\nabla^2 g_{oo} = kT g_{oo}. \tag{4.77}$$

Using the weak field approximation

$$g_{oo} = 1 - h_{oo} \sim 1 \tag{4.78}$$

for the symmetric metric, we obtain Carroll's Eq. (4.36) [2] (the Einstein field equation in the weak-field limit):

$$\nabla^2 h_{oo} = -kT g_{oo} = -kT_{oo}, \tag{4.79}$$

which is the Poisson equation (4.52) with $h_{oo} = -c^2\Phi/2, \quad k = 8\pi G/c^2, \quad T_{oo} = m/V$. Therefore the wave equation (4.28) is the eigenequation corresponding to the classical Einstein field equation. Einstein [4] arrived at the approximation (4.79) though an intermediate equation (Eq. (4.89b) of Ref. [4]):

$$\Box\gamma_{\mu\nu} = 2kT^*_{\mu\nu}, \quad T^*_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T, \tag{4.80}$$

in which the metric tensor was approximated by [4]:

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}. \tag{4.81}$$

Using the definition

$$T = g^{\mu\nu} T_{\mu\nu}, \tag{4.82}$$

an expression is obtained for $T_{\mu\nu}$ in terms of T:

$$g_{\mu\nu} T = (g_{\mu\nu} g^{\mu\nu}) T_{\mu\nu} = 4T_{\mu\nu}. \tag{4.83}$$

Equation (4.80) can therefore be written as the eigenequation

$$\left(\Box + \frac{1}{2} kT\right) g_{\mu\nu} = 0, \tag{4.84}$$

which is the wave quation (4.28) except for a factor (1/2) coming from the approximation method used by Einstein.

Using the weak-field limit of Eq. (4.33), we obtain

$$(\Box + km/V)\, q_0 = 0, \tag{4.85}$$

where $T = mc^2/V$ is again the rest energy density. Identifying q_0 as a scalar field [5] identifies Eq. (4.85) as the single-particle wave equation, which after quantization can be interpreted as the Klein-Gordon equation [5], whose wavefunction is identified with q_0 in the weak-field approximation. The Klein-Gordon equation is

$$\left(\Box + m^2c^2/\hbar^2\right) q_0 = 0, \tag{4.86}$$

so

$$E_0 = mc^2 = \frac{m^2c^4V}{\hbar^2 k}. \tag{4.87}$$

Equation (4.87) can be identified as the Planck/de Broglie postulate for any particle:

$$E_0 = \hbar\omega_0 = mc^2, \tag{4.88}$$

where ω_0 is the rest frequency of any particle. The rest frequency is defined by

$$\omega_0 = 8\pi c\ell^2/V, \tag{4.89}$$

where

$$\ell = (G\hbar/c^3)^{1/2} \tag{4.90}$$

is the Planck length. Equation (4.87) means that the product of the rest mass m and the rest volume V of any particle is a universal constant

$$mV = \hbar^2 k/c^2, \tag{4.91}$$

which is an important result of the generally covariant wave equation.

Using the operator equivalence of quantum mechanics [5]

$$p^\mu = i\hbar\partial^\mu,$$

$$p^\mu = (En/c, \boldsymbol{p}), \quad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \tag{4.92}$$

Eq. (4.86) becomes Einstein's equation of special relativity:

$$p^\mu p_\mu = \frac{En^2}{c^2} - p^2 = m^2c^2, \tag{4.93}$$

in which En denotes the total energy (kinetic plus potential) and mc^2 is the rest energy. From the equation [18]

$$\boldsymbol{F} = \gamma m\dot{\boldsymbol{v}} = \dot{\boldsymbol{p}}, \tag{4.94}$$

where $\boldsymbol{p} = \gamma m\boldsymbol{v}$ is the momentum in the limit of special relativity (weak-field limit), an expression is obtained for the kinetic energy in special relativity:

$$T = mc^2(\gamma - 1). \tag{4.95}$$

In the non-relativistic limit $v \ll c$, the Newtonian kinetic energy

$$T = \frac{1}{2}mv^2 \tag{4.96}$$

is obtained from the second Newton law, Eq. (4.73), which is self-consistently the non-relativistic weak-field limit of Eq. (4.33). Using the operator equivalence (4.92) in Eq. (4.96) gives the time-dependent free-particle Schrödinger equation [5,19]:

$$i\hbar\frac{\partial q_0}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}q_0. \tag{4.97}$$

Identifying the Hamiltonian operator as

$$H = -\frac{\hbar^2\nabla^2}{2m} \tag{4.98}$$

transforms Eq. (4.97) into the time-independent free-particle Schrödinger equation

$$Hq_0 = Tq_0, \tag{4.99}$$

which is a weak-field approximation to the wave equation (4.33) when we consider kinetic energy only [5]. The wavefunction of the Schrödinger equation (4.99) is [19]

$$q_0 = 1 + Ae^{iKZ} + Be^{-iKZ}, \tag{4.100}$$

which is the time-like component of the metric eigenfunction of Eq. (4.33) in the weak-field approximation used to recover Eq. (4.99).

These methods illustrate that wave or quantum mechanics can be considered to be an outcome of general relativity, and that the wave-function can be considered to be a deterministic property of general relativity, namely a metric four-vector, a metric tensor, or most generally a vielbein.

The first Newton law is obtained in the weak-field limit of the geodesic equation, or alternatively when the Christoffel symbol $\Gamma^0_{\nu 0}$ in Eq. (4.74) vanishes. These limits correspond to the flat spacetime in which there is no acceleration. Newton's law is contained within the conservation law for q_μ. The latter can be deduced from the Bianchi identity [2]

$$D^\mu G_{\mu\nu} := 0, \tag{4.101}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{4.102}$$

is the Einstein tensor. Noether's theorem gives the energy conservation law

$$D^\mu T_{\mu\nu} = 0, \tag{4.103}$$

and the metric compatibility assumption of standard general relativity [2] is

$$D^\rho g_{\mu\nu} = 0. \tag{4.104}$$

If we define [1]

$$R_{\mu\nu} := R_\mu q_\nu, \quad T_{\mu\nu} := T_\mu q_\nu, \quad g_{\mu\nu} = q_\mu q_\nu, \tag{4.105}$$

then the Bianchi identity (4.101) becomes

$$\begin{aligned} D^\mu G_{\mu\nu} &= (D^\mu R_\mu)q_\nu + R_\mu(D^\mu q_\nu) \\ &\quad -\tfrac{1}{2}Rq_\mu D^\mu q_\nu - \tfrac{1}{2}D^\mu(Rq_\mu)q_\nu \\ &:= 0. \end{aligned} \tag{4.106}$$

Using the metric compatibility assumption for q_ν, Eq. (4.32), gives the result

$$D^\mu G_\mu = 0, \quad G_\mu := R_\mu - \frac{1}{2}Rq_\mu. \tag{4.107}$$

This is the Bianchi identity for the field tensor:

$$G_\mu = kT_\mu. \tag{4.108}$$

Using Eq. (4.103) and the Leibniz theorem, the energy conservation law becomes

$$\begin{aligned} D^\mu(T_\mu q_\nu) &= (D^\mu T_\mu)q_\nu + T_\mu(D^\mu q_\nu) \\ &= (D^\mu T_\mu)q_\nu = 0, \end{aligned} \tag{4.109}$$

and the energy conservation law for T_μ is deduced to be

$$D^\mu T_\mu = 0. \tag{4.110}$$

The unified field equation (4.45) [1] becomes

$$D^\mu(G_\mu - kT_\mu) := 0. \tag{4.111}$$

Using the equations [1]:

$$R_\mu = \frac{1}{4}Rq_\mu, \tag{4.112}$$

both the energy conservation law (4.110) and the Bianchi identity (4.107) can be expressed as the equation

$$\left(D^\mu + \frac{1}{R}D^\mu R\right)q_\mu = 0. \tag{4.113}$$

4.5 Some Fundamental Equations Of Physics Derived From The Wave Equation

Equation (4.113) is similar in structure to a gauge transformation equation in generic gauge field theory [2,5,7-12]. On using the results [2]

$$D_\mu R = \partial_\mu R, \tag{4.114}$$

$$\begin{aligned} D_\mu q^\mu &= \partial_\mu q^\mu + \Gamma^\mu{}_{\mu\lambda}\, q^\lambda = \tfrac{1}{\sqrt{|q|}} \partial_\mu (\sqrt{|q|} q^\mu), \\ &= \partial_\mu q^\mu + \left(\tfrac{1}{\sqrt{|q|}} \partial_\mu \sqrt{|q|} \right) q^\mu, \end{aligned} \tag{4.115}$$

where $|q|$ is the modulus of the determinant of the symmetric metric $g_{\mu\nu} := q_\mu q_\nu$, Eq. (4.113) becomes

$$\left(\partial_\mu + \frac{1}{\sqrt{|q|}} \partial_\mu \sqrt{|q|} + \frac{1}{R} \partial_\mu R \right) q^\mu = 0, \tag{4.116}$$

a result which has been generated by the Leibniz theorem [2]

$$D_\mu (R q^\mu) = (D_\mu R) q^\mu + R(D_\mu q^\mu) = 0. \tag{4.117}$$

Now consider the definition of gauge transformation in generic gauge field theory [5]:

$$\psi' = S\psi \tag{4.118}$$

where ψ is the generic (n-dimensional) gauge field and S the rotation generator in n dimensions. Application of the Leibniz theorem produces

$$D_\mu (S\psi) = (D_\mu S)\psi + S(D_\mu \psi). \tag{4.119}$$

The covariant derivative in generic gauge field theory is defined through a vielbein [2], the generic gauge potential $A^a{}_\mu$, and a factor g (denoting generic charge):

$$D_\mu := \partial_\mu - igA^\mu, \quad A_\mu := m^a A^a{}_\mu; \tag{4.120}$$

and the gauge transformation (4.118) implies that

$$A'_\mu = A_\mu - \frac{i}{gS} \partial_\mu S, \tag{4.121}$$

i.e.,

$$igA'_\mu = igA_\mu + \frac{1}{S} \partial_\mu S. \tag{4.122}$$

The factor $-i$ in Eq. (4.120) originates in the fact that the gauge group generators m_a in generic gauge field theory are defined as imaginary-valued matrices. This procedure defines the upper index a of the vielbein $A^a{}_\mu$ [2]. However, a basis can always be found for the gauge group generators such that Eq. (4.120) becomes

$$D_\mu = \partial_\mu + gA_\mu. \tag{4.123}$$

Comparing Eqs. (4.123) and (4.116),

$$A_\mu = \frac{1}{g} \cdot \frac{1}{R} \partial_\mu R = \frac{\hbar}{e} \cdot \frac{1}{R} \partial_\mu R = B^{(0)} \partial_\mu R, \tag{4.124}$$

where $\hbar/e$ is the elementary unit of magnetic flux (the fluxon) and where $B^{(0)}$ is a magnetic flux density. Equation (4.124) combines the operator equivalence (4.92) of quantum mechanics with the minimal prescription ($p^\mu = eA^\mu$) in generic gauge field theory, giving the result

$$p^\mu = eA^\mu = i\hbar\partial^\mu. \tag{4.125}$$

This result has been obtained from the wave equation (4.33) and the Bianchi identity (4.107) in general relativity. Comparison of Eqs. (4.117) and (4.119) shows that the scalar curvature $R = -kT$ in general relativity plays the role of the rotation generator S in generic gauge field theory, and that the metric q_μ plays the role of the generic field A_μ or potential. The field can be a scalar field as in the single particle wave equation, Klein-Gordon, and Schrödinger equations (Sec. 4), but can also be a spinor, as in the Dirac equation, and a four-vector as in the Proca, d'Alembert, and Poisson equations of electrodynamics. In gravitation it has been shown in previous sections that the field can be a four-vector, a symmetric and anti-symmetric tensor and, most generally, a vielbein [2]. In $O(3)$ electrodynamics [7-12] the field or potential (Feynman's "universal influence" [5]) is the vielbein $A^a{}_\mu$, where the upper index denotes the Euclidean [2] complex circular basis ((1),(2),(3)) needed for the description of circular polarization in radiation. The lower index denotes non-Euclidean spacetime in general relativity. The upper index a is a basis for the tangent bundle of general relativity; and if we now make the *ansatz*

$$A^a{}_\mu = A^{(0)} q^a{}_\mu = A^{(0)} e^a{}_\mu, \tag{4.126}$$

we identify the internal index of a the field or potential or "universal influence" in gauge field theory with the basis index of the tangent space [2] in general relativity. This identification is the key to field unification in the new wave equation (4.25). In other words field unification is achieved by choosing the eigenfunction of the wave equation to represent the different fields that are presently thought to exist in nature: scalar fields, vector fields, symmetric or anti-symmetric tensor fields, spinor fields, and most generally, vielbeins. The weak field is a vielbein whose $SU(2)$ internal index describes the three massive weak field bosons [5], and the strong field is a vielbein whose $SU(3)$ internal index represents gluons. The internal index of the weak field therefore represents a physical tangent space of general relativity whose structure group is $SU(2)$, homomorphic with the structure group $O(3)$ of $O(3)$ electrodynamics [7-12]. The fiber bundle for both fields is therefore identified with the tangent bundle. In $O(3)$ electrodynamics the fibers are tied together with rotations in three dimensions represented by the structure group $SO(3)$ and the field is defined on this tangent or fiber bundle by the vielbein $A^a{}_\mu$. In weak field theory precisely the same procedure is followed, but the structure group becomes $SU(2)$ and the field becomes the vielbein $W^a{}_\mu$ whose three internal indices represent the three massive weak field bosons. In strong field theory the structure group is $SU(3)$ and the vielbein becomes $S^a{}_\mu$, where there are eight indices a [5]. The *ansatz* (4.126) therefore implies that the massive bosons of the weak field and the gluons of the strong field are different manifestations of the photons with indices (1), (2), and (3) of $O(3)$ electrodynamics. The $O(3)$ photons, the weak field bosons and the gluons are described by Eq. (4.25) in which the eigenfunctions are respectively ${A^a}_\mu, {W^a}_\mu$, and ${S^a}_\mu$, i.e., by the three wave equations of general relativity

$$(\square + kT){A^a}_\mu = 0, \tag{4.127}$$

$$(\square + kT){W^a}_\mu = 0, \tag{4.128}$$

$$(\square + kT){S^a}_\mu = 0. \tag{4.129}$$

In other words, the internal indices of the $O(3)$, weak and strong fields are different representations of the basis used to represent the tangent space in general relativity. The $O(3)$ electromagnetic field is represented by a vielbein in which the tangent space is defined in the $O(3)$ symmetry complex circular basis ((1),(2),(3)) [7-12]. This basis for the vielbein of the weak field becomes the three $SU(2)$ matrices (Pauli matrices), and there are eight $SU(3)$ symmetry matrices (geometrical generalizations [5] of the three complex two by two Pauli matrices to eight complex three by three matrices). These different basis representations are all representations of the same physical tangent space in general relativity.

In the currently accepted convention of the standard model and grand unified field theory the electromagnetic sector is represented by the field or potential A_μ in which there is no internal index, and the abstract fiber bundle of gauge field theory is not identified with the physical tangent bundle of general relativity. Consequently the standard model suffers from the inconsistencies described in the introduction, the most serious of these inconsistencies is that the Principle of General Relativity is not followed in the currently accepted convention known as "the standard model"–the Principle is applied to the gravitational field in the standard model but not to the electromagnetic, weak, and strong fields.

In $O(3)$ electrodynamics the *ansatz* (4.126) implies that

$$A_\mu = A^{(0)} q_\mu = \frac{1}{g} \cdot \frac{1}{R} \partial_\mu R \tag{4.130}$$

(where the scalar magnitude $A^{(0)}$ and the differential operator ∂_μ are the same for all three indices a). If for each index a we assume that

$$q_\mu = \frac{ds}{dx^\mu}, \tag{4.131}$$

then the *ansatz* (4.126) implies that

$$s = \frac{1}{gA^{(0)}} = \frac{1}{\kappa}. \tag{4.132}$$

For each index a the geodesic equation for $O(3)$ electrodynamics [7-12] becomes

$$\frac{d\kappa^\mu}{ds} + \Gamma^\mu{}_{\nu\sigma}\,\kappa^\nu\kappa^\sigma = 0, \quad \kappa^\mu = \frac{dq^\mu}{ds}, \tag{4.133}$$

an equation which defines the propagation, or path taken in non-Euclidean spacetime, of the three photons (1), (2), and (3) of $O(3)$ electrodynamics.

The wave equation (4.25) becomes the d'Alembert equation of $O(3)$ electrodynamics [7-12]

$$\Box A^a{}_\mu = -\frac{1}{\epsilon_0 c^2} j^a{}_\mu \tag{4.134}$$

if we define the four-current density by the vielbein

$$j^a{}_\mu = c^2 \epsilon_0 k T A^a{}_\mu . \tag{4.135}$$

Equation (4.134) represents three wave equations [7-12], one for each photon indexed (1), (2), and (3):

$$\Box A^{(1)}{}_\mu = -\tfrac{1}{\epsilon_0 c^2} j^{(1)}{}_\mu , \tag{4.136}$$

$$\Box A^{(2)}{}_\mu = -\tfrac{1}{\epsilon_0 c^2} j^{(2)}{}_\mu , \tag{4.137}$$

$$\Box A^{(3)}{}_\mu = -\tfrac{1}{\epsilon_0 c^2} j^{(3)}{}_\mu , \tag{4.138}$$

two transverse photons, (1) and (2), and one longitudinal (3). These three equations are evidently equations of general relativity, and are also gravitational wave equations multiplied on each side by the C negative scalar magnitude $A^{(0)}$. It follows from the foregoing discussion that these wave equations are also equations of the weak and strong fields with $A^a{}_\mu$ replaced respectively by $W^a{}_\mu$ and $S^a{}_\mu$. The weak field limit applied to Eq. (4.127) produces three Proca equations [5,7-12], one for each photon (i.e., for each index $a = (1), (2)$ and (3)):

$$\left(\Box + m^2 c^2/\hbar^2\right) A^{(i)}{}_\mu = 0, \quad i = 1, 2, 3, \tag{4.139}$$

and this procedure also produces the Planck/de Broglie postulate (4.95) applied to the photon, thus identifying the photon as a particle with mass. In the limit of electrostatics we obtain from Eq. (4.127) the Poisson equation

$$\nabla^2 A_0 = -R A_0 = k T A_0, \tag{4.140}$$

which shows that the source of the scalar potential A_0 is the scalar curvature R. This result appears to be an important indication of the fact that electric current can be obtained from the scalar curvature of the non-Euclidean spacetime, i.e., electromagnetic energy can be obtained from non-Euclidean spacetime through devices such as the motionless electromagnetic generator [12].

The identification of the $O(3)$ electromagnetic field as a vielbein implies that the unit vectors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}$ of the basis described by the upper Latin index a of the vielbein are orthonormal vectors of an Euclidean tangent space to the base manifold (non-Euclidean spacetime) described by the lower Greek index μ of the vielbein. The unit vectors define the $O(3)$ symmetry cyclic equations [7-12]:

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*}, \\ \boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} &= i\boldsymbol{e}^{(1)*}, \\ \boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} &= i\boldsymbol{e}^{(2)*}, \end{aligned} \tag{4.141}$$

and can be used to define a tangent at any point p of a curve in the non-Euclidean spacetime used to define the base manifold. The basis unit vectors are defined in terms of the Cartesian unit vectors of the tangent space by [7-12]

$$\begin{aligned} \boldsymbol{e}^{(1)} &= (1/\sqrt{2})(\boldsymbol{i} - i\boldsymbol{j}), \\ \boldsymbol{e}^{(2)} &= (1/\sqrt{2})(\boldsymbol{i} + i\boldsymbol{j}), \\ \boldsymbol{e}^{(3)} &= \boldsymbol{k}. \end{aligned} \tag{4.142}$$

It follows that the $O(3)$ electromagnetic field is defined in terms of the metric vectors:

$$\begin{aligned} \boldsymbol{A}^{(1)} &= A^{(0)}/\sqrt{2}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi} = A^{(0)}\boldsymbol{q}^{(1)}, \\ \boldsymbol{A}^{(2)} &= A^{(0)}/\sqrt{2}(\boldsymbol{i} + i\boldsymbol{j})e^{-i\phi} = A^{(0)}\boldsymbol{q}^{(2)}, \\ \boldsymbol{A}^{(3)} &= A^{(0)}\boldsymbol{k} = A^{(0)}\boldsymbol{q}^{(3)}, \end{aligned} \tag{4.143}$$

where ϕ is the electromagnetic phase. The unit vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ can be thought of as tangent vectors on a circle as illustrated in the following Argand diagram:

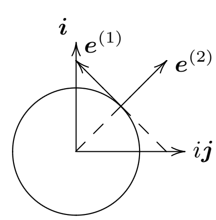

Fig. 4.1. Tangent Vectors

These tangent vectors are vectors of the tangent space to the base manifold. If we write

$$\begin{aligned} \boldsymbol{q}^{(1)'} &= \boldsymbol{q}^{(2)'} = \tfrac{1}{\sqrt{2}}(\boldsymbol{i}\cos\phi + \boldsymbol{j}\sin\phi), \\ \boldsymbol{q}^{(1)''} &= -\boldsymbol{q}^{(2)''} = \tfrac{1}{\sqrt{2}}(\boldsymbol{i}\sin\phi - \boldsymbol{j}\cos\phi), \end{aligned} \tag{4.144}$$

it can be seen in the following diagram that the metric vectors are tangent vectors that rotate around a circle for any given point Z:

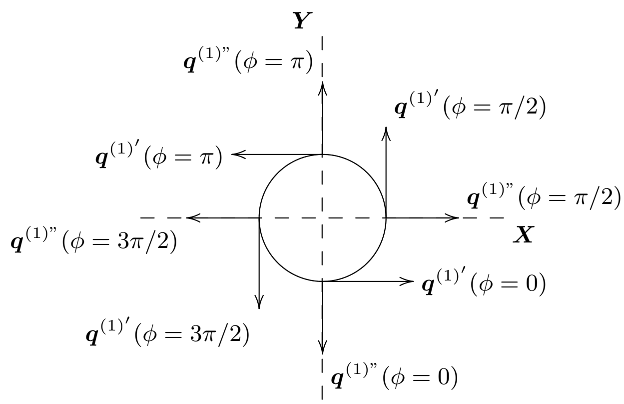

Fig. 4.2. Rotating Tangent Vectors

As we advance along the Z axis, which defines the unit vector $\boldsymbol{e}^{(3)}$ orthonormal to $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$, the path drawn out is a helix, and this is the geodesic (propagation path) for $O(3)$ radiation.

Having recognized that the $O(3)$ electromagnetic field is defined by the vielbein in Eq. (4.25), it becomes possible to define scalar-valued components of the electromagnetic field (and scalar fields in general) as scalar-valued vielbein components such as:

$$\begin{aligned} q^{(1)}{}_X &= (1/\sqrt{2})e^{i\phi}, \quad q^{(1)}{}_Y = -(1/\sqrt{2})e^{i\phi}, \\ q^{(2)}{}_X &= (1/\sqrt{2})e^{-i\phi}, \quad q^{(2)}{}_Y = (1/\sqrt{2})e^{-i\phi}, \\ q^{(3)}{}_Z &= 1. \end{aligned} \tag{4.145}$$

These scalar-valued vielbein components are components of the tangent-space vector:

$$\boldsymbol{q}_\mu = q^{(1)}{}_\mu \boldsymbol{e}^{(1)} + q^{(2)}{}_\mu \boldsymbol{e}^{(2)} + q^{(3)}{}_\mu \boldsymbol{e}^{(3)}, \tag{4.146}$$

which is defined by the three four-vectors [7-12] in the base manifold $q^{(1)}{}_\mu$, $q^{(2)}{}_\mu$, $q^{(3)}{}_\mu$, one four-vector for each index $a = (1), (2)$ and (3). The components of the $O(3)$ electromagnetic field are therefore

$$A^{(1)}{}_\mu = A^{(0)} q^{(1)}{}_\mu, \tag{4.147}$$

$$A^{(2)}{}_\mu = A^{(0)} q^{(2)}{}_\mu, \tag{4.148}$$

$$A^{(3)}{}_{\mu} = A^{(0)} q^{(3)}{}_{\mu}, \tag{4.149}$$

two transverse ($a = (1)$ and (2)) and one longitudinal ($a = (3)$).

The vielbein is well defined object in differential geometry [2] and can be used, for example, to generalize Riemann geometry through the Maurer-Cartan structure equations. The close similarity of vielbein theory to gauge theory is also well understood mathematically [2], but in the currently accepted convention of the standard model the vielbein has not been used because the identification of the fiber bundle of gauge field theory is the tangent bundle of general relativity has not been made. In this section we have identified the internal index of $O(3)$ electrodynamics with the tangent space of general relativity by identifying a with the indices $(1), (2)$, and (3). This identification allows results from vielbein theory and differential geometry to be used for unified field theory, i.e., both for general relativity and gauge theory. For example the $O(3)$ gauge field is defined by [2]:

$$\begin{aligned} G^{a}{}_{\mu\nu} &= (dA)^{a}{}_{\mu\nu} + (\omega \wedge A)^{a}{}_{\mu\nu} \\ &= \partial_\mu A^{a}{}_{\nu} - \partial_\nu A^{a}{}_{\mu} + \omega^{a}{}_{\mu b} A^{b}{}_{\nu} - \omega^{a}{}_{\nu b} A^{b}{}_{\mu}, \end{aligned} \tag{4.150}$$

which is a covariant exterior derivative in differential geometry. In Eq. (4.150) $\omega^{a}{}_{\mu b}$ is a spin affine connection. In gauge field theory the $O(3)$ electromagnetic gauge field is defined by the gauge-invariant commutator of covariant derivatives [5,7-12]:

$$\begin{aligned} G^{a}{}_{\mu\nu} = \tfrac{i}{g}[D_\mu, D_\nu] = \partial_\mu A^{a}{}_{\nu} - \partial_\nu A^{a}{}_{\mu} \\ + g(A^{b}{}_{\mu} A^{c}{}_{\nu} - A^{b}{}_{\nu} A^{c}{}_{\mu}). \end{aligned} \tag{4.151}$$

A comparison of Eq. (4.150) and (4.151) defines the spine affine connections in terms of the $O(3)$ fields or vector potentials:

$$\omega_{\mu}{}^{a}{}_{b} A^{b}_{\nu} - \omega_{\mu}{}^{a}{}_{b} A^{b}{}_{\mu} = g(A^{b}{}_{\mu} A^{c}{}_{\nu} - A^{b}{}_{\nu} A^{c}{}_{\mu}) = g\epsilon_{abc} A^{b}{}_{\mu} A^{c}{}_{\nu}. \tag{4.152}$$

Thus, the field or potential or "universal influence" $A^{a}{}_{\mu}$ has been defined in this section in terms of the scalar curvature in general relativity and also in terms of the spin affine connections. The gauge field $G^{a}{}_{\mu\nu}$ is invariant under the gauge transformation (4.128); i.e., if

$$A_\mu \rightarrow A_\mu - \frac{i}{g}\frac{1}{S}\partial_\mu S, \tag{4.153}$$

the gauge field is unchanged. This result is true for all four fields. In gravitation the equivalent of the gauge field is the Riemann tensor, which is covariant under coordinate transformation, while the Christoffel symbol is not covariant under coordinate transformation because it is not a tensor [2].

Some powerful results of vielbein theory may be translated directly into the language of unified field theory developed in this Letter, for example

$O(3)$ electrodynamics. The first of the Maurer-Cartan structure relations [2] of differential geometry is

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \tag{4.154}$$

and states that the covariant exterior derivative of the torsion form T^a (left hand side of Eq. (4.154) is the wedge product of the Riemann form $R^a{}_b$ and vielbein form e^b (right-hand side of Eq. (4.154)). Equation (4.154) is the inhomogeneous field equation of $O(3)$ electrodynamics:

$$D_\mu G^{\mu\nu,a} = \frac{1}{\mu_0} j^{\nu,a}, \tag{4.155}$$

where the charge-current density vielbein is defined by Eq. (4.135) of this Section. Equations (4.154) and (4.155) are equations of unified field theory– the torsion form T^a represents electromagnetism (or the weak and strong fields), and the Riemann form $R^a{}_b$ represents gravitation. In Ref. (1) the inhomogeneous equation (4.153) was inferred from Eq. (4.45) by multiplying it on both sides by the wedge $\wedge A^\mu{}_\nu$, and by defining the electromagnetic field tensor as

$$G_{\mu\nu} = G^{(0)}(R_\mu \wedge q_\nu - \frac{R}{2} q_\mu \wedge q_\nu) \tag{4.156}$$

and the charge current density as

$$j^\nu = \mu_0 G^{(0)} k D_\mu (T^\mu \wedge q^\nu). \tag{4.157}$$

The gravitational field and Riemann tensor were defined [1] by multiplying the novel field Eq. (4.45) on both sides by q_ν, so Eq. (4.45), the classical analogue of the wave equation (4.25) is an equation of unified field theory.

The second Maurer-Cartan structure relations is the Bianchi identity, and translates into the Bianchi identity of gravitation [2,5], and also into the identity (4.107) used in Sec. 4 to derive the gauge invariance equation (4.113). In $O(3)$ electrodynamics it becomes the homogeneous field equation [1], the Jacobi identity

$$D_\mu \widetilde{G}^{\mu\nu,a} := 0, \tag{4.158}$$

where $\widetilde{G}^{\mu\nu,a}$ is the dual [5,7-12] of $G^a{}_{\mu\nu}$.

The tetrad postulate of vielbein theory, Eq. (4.6), translates into the $O(3)$ symmetry cyclic relations

$$\begin{aligned}\partial_i \boldsymbol{q}^{(1)*}{}_j &= -i\kappa \boldsymbol{q}^{(2)}{}_i \times \boldsymbol{q}^{(3)}{}_j,\\ \partial_i \boldsymbol{q}^{(2)*}{}_j &= -i\kappa \boldsymbol{q}^{(3)}{}_i \times \boldsymbol{q}^{(1)}{}_j,\\ \partial_i \boldsymbol{q}^{(3)*}{}_i &= -i\kappa \boldsymbol{q}^{(1)}{}_i \times \boldsymbol{q}^{(2)}{}_j\end{aligned} \tag{4.159}$$

between space indices of the base manifold ($\mu = i = 1, 2, 3$).

In this section the $O(3)$ electromagnetic gauge field has been identified in three different ways: Eqs. (4.150), (4.151), and (4.156). Self-consistency demands that these three definitions by the same, giving Eq. (4.152), for example. This equation relates the spin affine connection and the vector potential. Comparing equations (4.151) and (4.156) gives the important result

$$G^{(0)}\left(R_\mu \wedge q_\nu - \frac{R}{2} q_\mu \wedge q_\nu\right) = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + g\epsilon_{abc} A^b{}_\mu A^c{}_\mu, \qquad (4.160)$$

which indicates that the group structure of generally covariant electrodynamics is non-Abelian and that generally covariant electrodynamics must be a gauge field theory with an internal gauge group such as $O(3)$, of higher symmetry than the conventional $U(1)$ of the standard model. The wedge product $R_\mu \wedge q_\nu$ is accordingly identified as

$$R_\mu \wedge q_\nu = \frac{1}{G^{(0)}}\left(\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu\right) \qquad (4.161)$$

and the wedge product $q_\mu \wedge q_\nu$ as

$$\frac{R}{2} q_\mu \wedge q_\nu = -\frac{g}{G^{(0)}} \epsilon_{abc} A^b{}_\mu A^c{}_\nu. \qquad (4.162)$$

If electrodynamics were a $U(1)$ Abelian theory, then the wedge product $q_\mu \wedge q_\nu$ would be zero:

$$q_\mu \wedge q_\nu = q_\mu q_\nu - q_\nu q_\mu = 0. \qquad (4.163)$$

The electromagnetic field would then disappear because [1]

$$R_\mu = \frac{1}{4} R q_\mu. \qquad (4.164)$$

The tetrad postulate in $U(1)$ symmetry gauge field theory would reduce to

$$D_\mu q_\nu = (\partial_\mu - igA_\mu) q_\nu = (\partial_\mu - igA^{(0)} q_\mu) q_\nu = 0. \qquad (4.165)$$

The $U(1)$ gauge field would then be

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = igA^{(0)2}(q_\mu q_\nu - q_\nu q_\mu) = 0 \qquad (4.166)$$

and would vanish, a result that is self consistent with Eq. (4.164). It is concluded that *general relativity implies higher symmetry electrodynamics*, a result that is crucial for the development of a unified field theory.

Finally in this section we use another important result of vielbein theory to derive the Dirac equation from the wave equation (4.25): the vielbein allows spinors to be developed in non-Abelian spacetime. Each component of the spinor must obey a Klein-Gordon equation (Ref. [5], p. 45). The Klein-Gordon equation is obtained from the wave equation (4.25) by considering the four scalar components of the vielbein (there are four such components for each index a). The solutions of the Dirac equation for a particle at rest are the positive and negative solutions, respectively,

$$\psi = u(0)\exp(-imt), \quad \psi = v(0)\exp(imt). \tag{4.167}$$

The two positive energy and two negative energy spinors in this limit become

$$u^{(1)}(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u^{(2)}(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v^{(1)}(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad v^{(2)}(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{4.168}$$

and these are identified as components of the vielbein. The Dirac equation has been obtained from the wave equation (4.25), which uses the vielbein as eigenfunction.

4.6 Discussion

The key to field unification (unification of general relativity and gauge theory) in this Letter is the realization that the internal index (fiber bundle index) of gauge theory is the tangent bundle index of general relativity. Fundamental geometry shows that this internal index is present in basic relations such as the one between Cartesian unit vectors in Euclidean spacetime, $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ in cyclic permutation. This internal index is implicitly assumed to exist in everyday geometry in flat space, but is the key to realizing that the most general eigenfunction for the wave equation (4.25) must be a vielbein. The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (or $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}$ of the complex circular basis) are most generally vielbeins. It follows in generally covariant electrodynamics that the field $A^a{}_\mu$ is also a vielbein and that the gauge group symmetry of electrodynamics must be $O(3)$ or higher. The existence of a $U(1)$ gauge field theory is prohibited by fundamental geometry, because in such a theory the internal index of the vielbein is missing. This is geometrically incorrect. These results are proven as follows.

Consider the displacement vector [1,14,15] in the three dimensions of Euclidean space:

$$\boldsymbol{r} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}. \tag{4.169}$$

The Cartesian unit vectors are

$$\boldsymbol{i} = \frac{\partial \boldsymbol{r}}{\partial X} \Big/ \left|\frac{\partial \boldsymbol{r}}{\partial X}\right|, \quad \boldsymbol{j} = \frac{\partial \boldsymbol{r}}{\partial Y} \Big/ \left|\frac{\partial \boldsymbol{r}}{\partial Y}\right|, \quad \boldsymbol{k} = \frac{\partial \boldsymbol{r}}{\partial Z} \Big/ \left|\frac{\partial \boldsymbol{r}}{\partial Z}\right|, \tag{4.170}$$

and the three metric vectors are [1,18,19]

$$\begin{aligned} \boldsymbol{q}_X &= \boldsymbol{q}^a(a=1) = \left|\tfrac{\partial \boldsymbol{r}}{\partial X}\right| \boldsymbol{i}, \\ \boldsymbol{q}_Y &= \boldsymbol{q}^a(a=2) = \left|\tfrac{\partial \boldsymbol{r}}{\partial Y}\right| \boldsymbol{j}, \\ \boldsymbol{q}_Z &= \boldsymbol{q}^a(a=3) = \left|\tfrac{\partial \boldsymbol{r}}{\partial Z}\right| \boldsymbol{k}. \end{aligned} \tag{4.171}$$

It follows that in Euclidean space that both the unit and metric vector components must be labeled with an upper and lower index:

$$\begin{aligned}
&q^1{}_1 = -1, \quad q^1{}_2 = 0, \quad q^1{}_3 = 0, \\
&q^2{}_1 = 0, \quad q^2{}_2 = -1, \quad q^2{}_3 = 0, \\
&q^3{}_1 = 0, \quad q^3{}_2 = 0, \quad q^3{}_3 = -1, \\
&q^1{}_1 = -i_1 = i_X = 1, \text{ etc.}
\end{aligned} \tag{4.172}$$

These results extend to Euclidean spacetime on using the index 0

$$q^0{}_0 = 1, \quad q^0{}_1 = 0, \quad q^0{}_2 = 0, \quad q^0{}_3 = 0. \tag{4.173}$$

Equations (172) and (4.173) define the vielbein $q^a{}_\mu$ where $a = 0, 1, 2, 3$, and $\mu = 0, 1, 2, 3$. More precisely, the vielbein is a vierbein or tetrad [2] because there are four internal or tangent space indices a and four indices μ of the base manifold. If the tetrad is used in the context of general relativity the a index becomes the tangent space index, and if the tetrad is used in gauge theory a is the index of the internal space that defines the gauge group. *Therefore fundamental geometry shows that the tetrad can be used both in general relativity and gauge theory, and this is the key to field unification.*

In Ref. [1] it has been shown that both the gravitational and electromagnetic field originate in Eq. (4.45): if the gravitational field is described through the symmetric metric tensor $q_\mu q_\nu$ then the electromagnetic field must be described through the anti-symmetric tensor:

$$G_{\mu\nu} = G^{(0)} \left(R_\mu \wedge q_\nu - \frac{1}{2} R q_\mu \wedge q_\nu \right). \tag{4.174}$$

This is again a result of geometry, essentially the result states that there exists a dot product between two vectors (symmetric metric tensor $q_\mu q_\nu$, used to describe the gravitational field) there must exist a cross product between the same two vectors (anti-symmetric metric tensor $q_\mu \wedge q_\nu$, used to describe the electromagnetic field). Taking the definition [1]

$$R = q^{\mu\nu(S)} R_{\mu\nu} = q^\mu q^\nu R_\mu q_\nu = -2 q^\mu R_\mu, \quad q^\nu q_\nu = -2, \tag{4.175}$$

it follows that

$$R_\mu = \frac{1}{4} R q_\mu, \quad G_{\mu\nu} = \frac{1}{4} G^{(0)} R \left(q_\mu \wedge q_\nu - 2 q_\mu \wedge q_\nu \right) \tag{4.176}$$

and that the electromagnetic field can be written in general as the wedge product:

$$G_{\mu\nu} = -\frac{1}{4} G^{(0)} R (q_\mu \wedge q_\nu). \tag{4.177}$$

The minus sign in Eq. (4.177) is a matter of convention and so the electromagnetic field can be succinctly expressed, within a factor $B^{(0)}$, as the wedge product of q_μ and q_ν:

$$G_{\mu\nu} = B^{(0)}(q_\mu \wedge q_\nu), \tag{4.178}$$

where $B^{(0)}$ has the units of magnetic flux density [7-12]:

$$B^{(0)} = \frac{1}{4} G^{(0)} R. \tag{4.179}$$

The wedge product of two one forms in differential geometry is defined [2] by

$$A_\mu \wedge B_\nu = (A \wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu. \tag{4.180}$$

Therefore the wedge product vanishes if q_μ and q_ν are considered as four vectors with no internal index. *It follows that electromagnetism cannot be a gauge theory with no internal index, and therefore cannot be a $U(1)$ gauge field theory.* The wedge product of the two vielbeins $q^a{}_\mu$ and $q^b{}_\nu$ is

$$(q^a \wedge q^b)_{\mu\nu} = q^a{}_\mu q^b{}_\nu - q^a{}_\nu q^b{}_\mu, \tag{4.181}$$

and the electromagnetic field is the differential two-form

$$G^c{}_{\mu\nu} = B^{(0)}(q^a \wedge q^b)_{\mu\nu}. \tag{4.182}$$

In differential geometry, the Greek indices become redundant (i.e., can be assumed implicitly to be always the same on the left and right hand sides of an equation in differential geometry, the theory of differential forms), so the Greek indices can be suppressed [2]. Equation (4.182) can therefore be written as

$$G^c = B^{(0)} q^a \wedge q^b, \tag{4.183}$$

and, within a factor $B^{(0)}$, the electromagnetic field is a torsion two form T^c:

$$G^c = B^{(0)} T^c = B^{(0)} q^a \wedge q^b. \tag{4.184}$$

The first Maurer-Cartan structure relation (Eq. (4.154)) relates the torsion two form to the Riemann form, and so the first Maurer-Cartan structure relation becomes a relation between gravitation (Riemann form) and electromagnetism (torsion form). By adjusting the index a on the torsion form, the Maurer-Cartan structure relation becomes one between the weak field and gravitation, and the strong field and gravitation. This inter-relation between fields is a result of geometry and of the novel grand unified field theory developed in this Letter.

In the language of tetrads and wedge products the geometrical equation $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ becomes

$$(q^1 \wedge q^2)_{12} = q^1{}_1 \wedge q^2{}_2 = q^1{}_1 q^2{}_2 - q^1{}_2 q^2{}_1 = q^1{}_1 q^2{}_2 = (q^3)_{12} = q^3{}_3. \tag{4.185}$$

in Euclidean spacetime in the Cartesian basis the tetrad is non-zero if and only if $a = \mu_1$ so it has been implicitly assumed that $q^a{}_\mu$ can be written as q_μ. This assumption means that the existence of the internal index a in basic geometry has been overlooked. In gauge theory this has led to the incorrect assumption that there can exist a gauge theory (electromagnetism) with no internal index. Careful consideration shows however that the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the following tetrad components:

$$-\boldsymbol{i} := (0, q^1{}_1, 0, 0), \quad -\boldsymbol{j} := (0, 0, q^2{}_2, 0), \quad -\boldsymbol{k} := (0, 0, 0, q^3{}_3). \tag{4.186}$$

In a non-Euclidean space (base manifold) defined [1,14,15] by the curvilinear coordinate basis (u_1, u_2, u_3) the unit vectors are the orthonormal tangent space vectors

$$\boldsymbol{e}^a = \frac{\partial \boldsymbol{r}}{\partial u^a} / \left| \frac{\partial \boldsymbol{r}}{\partial u^a} \right|, \quad a = 1, 2, 3, \tag{4.187}$$

obeying the $O(3)$ cyclic relations

$$\boldsymbol{e}^1 \times \boldsymbol{e}^2 = \boldsymbol{e}^3, \quad \text{et cyclicum}, \tag{4.188}$$

and the metric vectors are

$$\boldsymbol{q}^a = \frac{\partial \boldsymbol{r}}{\partial u^a}, \quad a = 1, 2, 3, \tag{4.189}$$

i.e., the tetrad components

$$q^a{}_1 = -\frac{\partial X}{\partial u^a}, \quad q^a{}_2 = -\frac{\partial Y}{\partial u^a}, \quad q^a{}_3 = -\frac{\partial Z}{\partial u^a}. \tag{4.190}$$

The tetrad in four-dimensional spacetime is therefore $q^a{}_\mu$. The upper index a of the tetrad denotes a flat, orthonormal tangent spacetime, and the lower index μ the non-Euclidean base manifold (the non-Euclidean spacetime of general relativity). The structure factors [1] are:

$$h^a = (q^a{}_0 q^{a0} - q^a{}_1 q^{a1} - q^a{}_2 q^{a2} - q^a{}_3 q^{q3}). \tag{4.191}$$

In general relativity the metric $q^a{}_\mu$ always has an upper index a, and a lower index μ, and the tetrad $q^a{}_\mu$ is the eigenfunction of the wave equation (4.25) of grand unified field theory. It has been demonstrated in this Letter that this wave equation is the direct result of the tetrad postulate, Eq. (4.6), and so is the direct result of geometry. More generally, it has also been demonstrated in this Letter that there must exist an internal index a in all geometrical relations, such as the relation between Cartesian unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

$O(3)$ electrodynamics [7-12] is therefore Eq. (4.178) when the internal index a is (1), (2), and (3), and $O(3)$ *electrodynamics is the direct result of general relativity, and of geometry.* In other words the very existence of gravitation is empirical evidence for the existence of $O(3)$ electrodynamics, because gravitation is described through $q^a{}_\mu q^b{}_\nu$ and $O(3)$ electrodynamics by $q^a{}_\mu \wedge q^b{}_\nu$. Both fields originate in the classical equation (4.45) [1], which is the classical limit of the wave equation (4.25). In $O(3)$ electrodynamics the tetrad postulate (6) becomes the cyclic equations with $O(3)$ symmetry:

$$\begin{aligned}
\partial_\mu \boldsymbol{A}^{(3)*}{}_\nu &= -ig\boldsymbol{A}^{(1)}{}_\mu \times \boldsymbol{A}^{(2)}{}_\nu, \\
\partial_\mu \boldsymbol{A}^{(1)*}{}_\nu &= -ig\boldsymbol{A}^{(2)}{}_\mu \times \boldsymbol{A}^{(3)}{}_\nu, \\
\partial_\mu \boldsymbol{A}^{(2)*}{}_\nu &= -ig\boldsymbol{A}^{(3)}{}_\mu \times \boldsymbol{A}^{(1)}{}_\nu,
\end{aligned} \tag{4.192}$$

where we have used the relation $A^a{}_\mu = A^{(0)} q^a{}_\mu$. The tetrad postulate (4.6) shows that:

$$\begin{aligned}
\partial_\mu \boldsymbol{A}^{(1)*}{}_\nu - \partial_\nu \boldsymbol{A}^{(1)*}{}_\mu &= -ig\boldsymbol{A}^{(2)}{}_\mu \times \boldsymbol{A}^{(3)}{}_\nu, \\
\partial_\mu \boldsymbol{A}^{(2)*}{}_\nu - \partial_\nu \boldsymbol{A}^{(2)*}{}_\mu &= -ig\boldsymbol{A}^{(3)}{}_\mu \times \boldsymbol{A}^{(1)}{}_\nu, \\
\boldsymbol{B}^{(3)*}{}_{\mu\nu} &= -ig\boldsymbol{A}^{(1)}{}_\mu \times \boldsymbol{A}^{(2)}{}_\nu.
\end{aligned} \tag{4.193}$$

The gauge field in $O(3)$ electrodynamics is defined by the cyclic relations [7-12]

$$\begin{aligned}
\boldsymbol{G}^{(3)*}{}_{\mu\nu} &= \partial_\mu \boldsymbol{A}^{(3)*}{}_\nu - \partial_\nu \boldsymbol{A}^{(3)*}{}_\mu - ig\boldsymbol{A}^{(1)}{}_\mu \times \boldsymbol{A}^{(2)}{}_\nu, \\
\boldsymbol{G}^{(1)*}{}_{\mu\nu} &= \partial_\mu \boldsymbol{A}^{(1)*}{}_\nu - \partial_\nu \boldsymbol{A}^{(1)*}{}_\mu - ig\boldsymbol{A}^{(2)}{}_\mu \times \boldsymbol{A}^{(3)}{}_\nu, \\
\boldsymbol{G}^{(2)*}{}_{\mu\nu} &= \partial_\mu \boldsymbol{A}^{(2)*}{}_\nu - \partial_\nu \boldsymbol{A}^{(2)*}{}_\mu - ig\boldsymbol{A}^{(3)}{}_\mu \times \boldsymbol{A}^{(1)}{}_\nu.
\end{aligned} \tag{4.194}$$

But we know from Eq. (4.178) that

$$\begin{aligned}
\boldsymbol{G}^{(3)*}{}_{\mu\nu} &= -iB^{(0)}\boldsymbol{q}^{(1)}{}_\mu \times \boldsymbol{q}^{(2)}{}_\nu, \\
\boldsymbol{G}^{(1)*}{}_{\mu\nu} &= -iB^{(0)}\boldsymbol{q}^{(2)}{}_\mu \times \boldsymbol{q}^{(3)}{}_\nu, \\
\boldsymbol{G}^{(2)*}{}_{\mu\nu} &= -iB^{(0)}\boldsymbol{q}^{(3)}{}_\mu \times \boldsymbol{q}^{(1)}{}_\nu,
\end{aligned} \tag{4.195}$$

so in $O(3)$ electrodynamics there exist the following three fundamental relations:

$$g\boldsymbol{A}^{(1)}{}_\mu \times \boldsymbol{A}^{(2)}{}_\nu = B^{(0)}\boldsymbol{q}^{(1)}{}_\mu \times \boldsymbol{q}^{(2)}{}_\nu, \quad \text{et cyclicum.} \tag{4.196}$$

Finally the realization that the electromagnetic field must be a tetrad allows the description of the internal space by any appropriate index of the orthonormal tangent space, for example a can be (1),(2),(3) of the complex circular basis, or it can be (X, Y, Z) of the Cartesian basis. So $O(3)$ electrodynamics, or any higher symmetry electrodynamics, can be developed using any well defined index a of the tangent space of general relativity. This means that electrodynamics can be developed as an $SU(2)$ symmetry gauge field theory, or as an $SU(3)$ symmetry gauge field symmetry. This suggests that the weak and strong fields may both be manifestations of the electromagnetic field. Essentially, one field is changed into another by changing the index a. Therefore there emerge many possible inter-relations between fields once it is realized that the index a is always present in the tetrad $q^a{}_\mu$, i.e., in the eigenfunction of the wave equation (4.25).

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