# On Connections of the Anti-Symmetric and Totally Anti-Symmetric Torsion Tensor

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## Abstract

Based on the compatibility of the metric only, in general, it is shown that the Christoffel connection $\Gamma^{\rho}{}_{\mu\nu}$ is not anti-symmetric in the last two indices. The elements of the Christoffel connection can be computed from the tetrad postulate but are not uniquely defined. The same holds for Christoffel symbols of a totally anti-symmetric torsion tensor. Non-diagonal elements of the metric of spacetime have to be restricted to a special form to allow fulfillment of the tetrad postulate. By additional constraints it is possible to obtain Christoffel connections being anti-symmetric in non-diagonal terms, but with non-zero diagonal elements. It is also shown that for a totally antisymmetric torsion, if the connection is antisymmetric in any two indices, then it is totally antisymmetric.

## 1. Introduction

In this paper we investigate the symmetry of the Christoffel symbols. It has been shown in various papers of ECE theory [1]-[3] that the symbols cannot be symmetric in their two last indices. This was assumed in Einsteinian relativity but leads to inescapable contradictions. For example the Bianchi identity does not result in a zero sum for the curvature terms but exhibits derivatives of torsion terms at the right hand side. Therefore curvature is inseparable from torsion in general. The most general form of the Christoffel symbols then is asymmetric in their last two indices. Special cases will be considered in this paper. In Cartan geometry the torsion tensor is defined as anti-symmetric in its two last indices. As an additional constraint, the totally anti-symmetric torsion tensor has the following anti-symmetries:

$$\mathrm{T}^{\rho}{}_{\mu\nu} = -\mathrm{T}^{\rho}{}_{\nu\mu} \tag{1}$$

$$\mathrm{T}^{\rho}{}_{\mu\nu} = -\mathrm{T}^{\mu}{}_{\rho\nu} \tag{2}$$

$$\mathrm{T}^{\rho}{}_{\mu\nu} = -\mathrm{T}^{\nu}{}_{\mu\rho} \tag{3}$$

Any two of the anti-symmetries automatically imply the third. For example, if we assume the anti-symmetries of equations (1) and (2), then

$$T^{\nu}{}_{\mu\rho} = -T^{\nu}{}_{\rho\mu} = T^{\rho}{}_{\nu\mu} = -T^{\rho}{}_{\mu\nu} \; . \tag{4}$$

This is true for any two of the three anti-symmetries.

## 2. Derivation of Jensen's Expression for the Christoffel Symbols with Torsion

The well known relation between the symmetric Christoffel symbols and the metric is [4]

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right) . \tag{5}$$

Now assume connections of arbitrary symmetry. Then this formula is not valid and we have to start from the equations of metric compatibility as for example in [5]:

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \Gamma^{\lambda}{}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\rho\nu} g_{\mu\lambda} = 0, \tag{6}$$

$$\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \Gamma^{\lambda}{}_{\mu\upsilon} g_{\lambda\varrho} - \Gamma^{\lambda}{}_{\mu\rho} g_{\nu\lambda} = 0, \tag{7}$$

$$\frac{\partial g_{\rho\mu}}{\partial x^{\upsilon}} - \Gamma^{\lambda}{}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}{}_{\nu\mu} g_{\rho\lambda} = 0. \tag{8}$$

Subtract (7) and (8) from (6):

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}{}_{\rho\mu} g_{\lambda\nu} + \Gamma^{\lambda}{}_{\mu\nu} g_{\lambda\rho} + \Gamma^{\lambda}{}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}{}_{\rho\nu} g_{\mu\lambda} + \Gamma^{\lambda}{}_{\mu\rho} g_{\nu\lambda} + \Gamma^{\lambda}{}_{\nu\mu} g_{\rho\lambda} = 0. \tag{9}$$

Apply definition of torsion:

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + T^{\lambda}{}_{\mu\rho} g_{\lambda\nu} + T^{\lambda}{}_{\nu\rho} g_{\lambda\mu} + (\Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\lambda}{}_{\nu\mu}) g_{\rho\lambda} = 0. \tag{10}$$

Add $-2\Gamma^{\lambda}{}_{\mu\nu} g_{\rho\lambda}$ on both sides and apply definition of torsion again:

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + T^{\lambda}{}_{\mu\rho} g_{\lambda\nu} + T^{\lambda}{}_{\nu\rho} g_{\lambda\mu} - T^{\lambda}{}_{\mu\nu} g_{\rho\lambda} = -2\Gamma^{\lambda}{}_{\mu\nu} g_{\rho\lambda} \; . \tag{11}$$

Now evaluate the pull down of indexes by the metric:

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + T_{\nu\mu\rho} + T_{\mu\nu\rho} - T_{\rho\mu\nu} = -2\Gamma_{\rho\mu\nu} \; . \tag{12}$$

Raise the first index of $\Gamma_{\rho\mu\nu}$ by multiplying with $g^{\rho\sigma}$ and multiply by $-1$:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} - T_{\nu\mu\rho} - T_{\mu\nu\rho} + T_{\rho\mu\nu} \right) \quad . \tag{13}$$

This is the result of Carroll [5] with additional torsion terms. The corresponding equation of Jensen [6] reads:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} - T_{\mu\nu\sigma} - T_{\nu\mu\sigma} + T_{\sigma\mu\nu} \right) \quad . \tag{14}$$

This is identical with the result (13) by interchanging the indices ρ and σ.

## 3. Consequences of Anti-Symmetric Torsion

The Einstein summation convention will not be used in the following discussion. Later it will be seen that repetitive indices will appear where summation is not implied, imparting possible confusion.

Inverting $\mu$ and $\nu$ in equation (14) gives

$$\Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2} \sum_{\sigma} g^{\rho\sigma} \left( \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}} - T_{\nu\mu\sigma} - T_{\mu\nu\sigma} + T_{\sigma\nu\mu} \right) . \tag{15}$$

Adding (14) and (15) results in

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2} \sum_{\sigma} g^{\rho\sigma} \left( \frac{\partial (g_{\sigma\nu} + g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial (g_{\mu\sigma} + g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial (g_{\mu\nu} + \partial g_{\nu\mu})}{\partial x^{\sigma}} - T_{\mu\nu\sigma} - T_{\nu\mu\sigma} + T_{\sigma\mu\nu} - T_{\nu\mu\sigma} - T_{\mu\nu\sigma} + T_{\sigma\nu\mu} \right) \tag{16a}$$

which reduces to

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2} \sum_{\sigma} g^{\rho\sigma} \left( \frac{\partial (g_{\sigma\nu} + g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial (g_{\mu\sigma} + g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial (g_{\mu\nu} + \partial g_{\nu\mu})}{\partial x^{\sigma}} - 2T_{\mu\nu\sigma} - 2T_{\nu\mu\sigma} + T_{\sigma\mu\nu} + T_{\sigma\nu\mu} \right) . \tag{16b}$$

Subtracting (15) from (14) gives

$$\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2} \sum_{\sigma} g^{\rho\sigma} \left( \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\nu\mu}}{\partial x^{\sigma}} - T_{\mu\nu\sigma} - T_{\nu\mu\sigma} + T_{\sigma\mu\nu} + T_{\nu\mu\sigma} + T_{\mu\nu\sigma} - T_{\sigma\nu\mu} \right) \tag{16c}$$

which upon collecting like terms becomes

$$\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2} \sum_{\sigma} g^{\rho\sigma} \left( \frac{\partial (g_{\sigma\nu} - g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial (g_{\mu\sigma} - g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial (g_{\mu\nu} - \partial g_{\nu\mu})}{\partial x^{\sigma}} + T_{\sigma\mu\nu} - T_{\sigma\nu\mu} \right) . \tag{17}$$

Torsion is given by [5],

$$\mathrm{T}^{c}{}_{ab} = \Gamma^{c}{}_{ab} - \Gamma^{c}{}_{ba} \,, \tag{18}$$

and is easily seen to be anti-symmetric in indices $a$ and $b$.

Noting that

$$\mathrm{T}^{c}{}_{\mu\nu} = \mathrm{T}^{c}{}_{ab} q^{a}_{\ \mu}\, q^{b}_{\ \nu} \tag{19}$$

where $q^{a}_{\ \mu}$ and $q^{b}_{\ \nu}$ are elements of the Cartan tetrad, if one applies equation (19) to equation (18), it is seen that the tensor $\mathrm{T}^{c}{}_{\mu\nu}$ is anti-symmetric in $\mu$ and $\nu$. Further, since

$$\mathrm{T}^{\lambda}{}_{\mu\nu} = \mathrm{T}^{c}{}_{\mu\nu} q^{\lambda}_{\ c} \,, \tag{20}$$

we see that $\mathrm{T}^{\lambda}{}_{\mu\nu}$ is anti-symmetric in $\mu$ and $\nu$. Writing

$$T_{\sigma\mu\nu} = \mathrm{T}^{\lambda}{}_{\mu\nu} g_{\lambda\sigma} \tag{21}$$

the tensor $T_{\sigma\mu\nu}$ is also seen to be anti-symmetric in $\mu$ and $\nu$ , i.e.

$$T_{\sigma\mu\nu} = -T_{\sigma\nu\mu}. \tag{22}$$

With this anti-symmetry, equation (17) becomes

$$\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu} - g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial(g_{\mu\sigma} - g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu} - \partial g_{\nu\mu})}{\partial x^{\sigma}} + 2T_{\sigma\mu\nu}\right). \tag{23}$$

Since the metric is symmetric, equation (23) simplifies to the definition of torsion

$$\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} = \sum_{\sigma} g^{\rho\sigma}\, T_{\sigma\mu\nu} = \mathrm{T}^{\rho}{}_{\mu\nu} \tag{24}$$

as expected. If the metric were not symmetric, this would not be compatible with the definition of torsion.

Noting the anti-symmetry of equation (22), equation (16b) becomes

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu} + g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial(g_{\mu\sigma} + g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu} + \partial g_{\nu\mu})}{\partial x^{\sigma}} - 2T_{\mu\nu\sigma} - 2T_{\nu\mu\sigma}\right). \tag{25}$$

Noting

$$T_{\mu\nu\sigma} = \mathrm{T}^{\lambda}{}_{\nu\sigma} g_{\lambda\mu} = \left(\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\sigma\nu}\right) g_{\lambda\mu} \,, \tag{26}$$

$$T_{\nu\mu\sigma} = \mathrm{T}^{\lambda}{}_{\mu\sigma} g_{\lambda\nu} = \left(\Gamma^{\lambda}{}_{\mu\sigma} - \Gamma^{\lambda}{}_{\sigma\mu}\right) g_{\lambda\nu} \,, \tag{27}$$

equation (25) becomes

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu}+g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial(g_{\mu\sigma}+g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu}+\partial g_{\nu\mu})}{\partial x^{\sigma}} - 2\left(\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\sigma\nu}\right)g_{\lambda\mu} \; - 2\left(\Gamma^{\lambda}{}_{\mu\sigma} - \Gamma^{\lambda}{}_{\sigma\mu}\right)g_{\lambda\nu} \;\right) \tag{28}$$

If we expand the summation, this becomes

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu}+g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial(g_{\mu\sigma}+g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu}+\partial g_{\nu\mu})}{\partial x^{\sigma}}\right) - \left(\Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\mu}{}^{\rho}{}_{\nu}\right) - \left(\Gamma_{\nu\mu}{}^{\rho} - \Gamma_{\nu}{}^{\rho}{}_{\mu}\right) \tag{29}$$

which re-arranges to

$$\left(\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu}\right) + \left(\Gamma_{\mu\nu}{}^{\rho} + \Gamma_{\nu\mu}{}^{\rho}\right) - \left(\Gamma_{\mu}{}^{\rho}{}_{\nu} + \Gamma_{\nu}{}^{\rho}{}_{\mu}\right) = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu}+g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial(g_{\mu\sigma}+g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu}+\partial g_{\nu\mu})}{\partial x^{\sigma}}\right). \tag{30}$$

Since the metric of space-time is symmetric, equation (30) reduces to

$$\left(\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu}\right) + \left(\Gamma_{\mu\nu}{}^{\rho} + \Gamma_{\nu\mu}{}^{\rho}\right) - \left(\Gamma_{\mu}{}^{\rho}{}_{\nu} + \Gamma_{\nu}{}^{\rho}{}_{\mu}\right) = \sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{31}$$

This is the generalization of (5) including torsion.

## 4. Totally Anti-symmetric Torsion

Suppose that, in addition to the anti-symmetry of equation (22), we assume

$$\mathrm{T}^{\rho}{}_{\mu\nu} = -\mathrm{T}^{\mu}{}_{\rho\nu} \tag{32}$$

or equivalently

$$T_{\rho\mu\nu} = -T_{\mu\rho\nu} \;. \tag{33}$$

Such a torsion tensor is termed totally anti-symmetric. Now equation (25) becomes

$$\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu}+g_{\nu\sigma})}{\partial x^{\mu}} + \frac{\partial(g_{\mu\sigma}+g_{\sigma\mu})}{\partial x^{\nu}} - \frac{\partial(g_{\mu\nu}+\partial g_{\nu\mu})}{\partial x^{\sigma}}\right). \tag{34}$$

Comparing equations (30) and (34) we see that

$$\left(\Gamma_{\mu\nu}{}^{\rho} + \Gamma_{\nu\mu}{}^{\rho}\right) = \left(\Gamma_{\mu}{}^{\rho}{}_{\nu} + \Gamma_{\nu}{}^{\rho}{}_{\mu}\right). \tag{35}$$

To the authors' knowledge, this is a new constraint equation on the Christoffel connection for a totally anti-symmetric torsion tensor.

Note that equation (35) can be written

$$\sum_\lambda g^{\rho\lambda}\left(\left(\Gamma_{\mu\nu\lambda}+\Gamma_{\nu\mu\lambda}\right)-\left(\Gamma_{\mu\lambda\nu}+\Gamma_{\nu\lambda\mu}\right)\right)=0 \tag{36}$$

If for example, the connection can be shown to be antisymmetric in a pair of indices such as

$$\Gamma^\rho{}_{\mu\nu}+\Gamma^\rho{}_{\nu\mu}=0 \tag{37}$$

then

$$\sum_\lambda g^{\rho\lambda}\left(\left(\Gamma_{\mu\nu\lambda}+\Gamma_{\nu\mu\lambda}\right)+\left(\Gamma_{\mu\nu\lambda}+\Gamma_{\nu\mu\lambda}\right)\right)=0 \tag{38}$$

or

$$\sum_\lambda g^{\rho\lambda}\left(\Gamma_{\mu\nu\lambda}+\Gamma_{\nu\mu\lambda}\right)=0 \tag{39}$$

From this we have

$$\Gamma_{\mu\nu}{}^\rho=-\Gamma_{\nu\mu}{}^\rho \tag{40}$$

Then from equation (35)

$$\Gamma_\mu{}^\rho{}_\nu=-\Gamma_\nu{}^\rho{}_\mu \tag{41}$$

That is, if the torsion is totally antisymmetric and the connection is antisymmetric in any two indices, then the connection is totally antisymmetric.

Without restriction on the connection, since the metric is symmetric, equation (34) becomes

$$\Gamma^\rho{}_{\mu\nu}+\Gamma^\rho{}_{\nu\mu}=\sum_\sigma g^{\rho\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu}+\frac{\partial g_{\mu\sigma}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{42}$$

We note in passing that the Christoffel connection without torsion given by Carroll [5] and Wald [4] is a special case of this equation for zero torsion when

$$\Gamma^\rho{}_{\mu\nu}=\Gamma^\rho{}_{\nu\mu}. \tag{43}$$

In general, without further arguments than metric compatibility, for a totally anti-symmetric torsion, equation (42) implies some degree of antisymmetry for the connection, i.e. when

- the metric is constant then the connection is totally antisymmetric

- the metric is anti-symmetric, then

$$\sum_\sigma g^{\rho\sigma}\left(\frac{\partial(g_{\sigma\nu}+g_{\nu\sigma})}{\partial x^\mu}+\frac{\partial(g_{\mu\sigma}+g_{\sigma\mu})}{\partial x^\nu}-\frac{\partial(g_{\mu\nu}+\partial g_{\nu\mu})}{\partial x^\sigma}\right)=0 \qquad (44)$$

  The connection is again totally antisymmetric. As explained earlier, this is only a hypothetical case.
- the metric is diagonal, the connection is, without other considerations, totally antisymmetric except for the diagonal elements.

## 5. Totally Antisymmetric Torsion with Diagonal Metric

Consider now, the case where the metric has diagonal components only. Given this, equation (34) reduces to

$$\Gamma^\rho{}_{\mu\nu}+\Gamma^\rho{}_{\nu\mu}=\begin{cases} g^{\rho\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^\mu}+\frac{\partial g_{\mu\rho}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) & if\ \ \rho=\nu\ or\ \rho=\mu\ \ or\ \ \mu=\nu \\ 0 & otherwise \end{cases} \qquad (45)$$

That is, on the basis of the compatibility of the metric alone, for a metric that is diagonal and torsion is totally antisymmetric, $\Gamma^\rho{}_{\mu\nu}$ is either "totally antisymmetric" or zero, with the exception of the diagonals, as indicated in equation (45). The diagonal elements are given by

$$2\Gamma^\rho{}_{\mu\mu}=-g^{\rho\rho}\frac{\partial g_{\mu\mu}}{\partial x^\rho} \qquad \rho\neq\nu,\ \rho\neq\mu\,,\mu=\nu \qquad (46)$$

$$\Gamma^\nu{}_{\mu\nu}+\Gamma^\nu{}_{\nu\mu}=g^{\nu\nu}\frac{\partial g_{\nu\nu}}{\partial x^\mu} \qquad \rho=\nu,\ \rho\neq\mu\,,\mu\neq\nu \qquad (47)$$

$$\Gamma^\mu{}_{\mu\nu}+\Gamma^\mu{}_{\nu\mu}=g^{\mu\mu}\frac{\partial g_{\mu\mu}}{\partial x^\nu} \qquad \rho\neq\nu,\ \rho=\mu\,,\mu\neq\nu \qquad (48)$$

$$2\Gamma^\mu{}_{\mu\mu}=g^{\mu\mu}\frac{\partial g_{\mu\mu}}{\partial x^\mu} \qquad \rho=\nu,\ \rho=\mu\,,\mu=\nu \qquad (49)$$

This argument alone makes symmetry of the connection impossible unless it is identically zero. Other arguments are needed if totally antisymmetry of the connection is to be proven. Recall that there is no summation on repeated indices implied in these equations.

## 6. Computation schemes for connections with torsion

Taking one of the equations (6-8), we obtain a linear equation system for $4^3 = 64$ components of the $\Gamma$ connections, assuming no further symmetry. At the same time we have 64 single equations for all combinations of ρ, μ, ν. This could give a well defined equation set for determining all Γ connections. However, the metric is symmetric, introducing certain symmetry into the equations so that not all equations are independent of each other. Using a diagonal metric, where each element depends on all four coordinates, for example

$$g = \begin{pmatrix} A(t,r,\varphi,\theta) & 0 & 0 & 0 \\ 0 & B(t,r,\varphi,\theta) & 0 & 0 \\ 0 & 0 & C(t,r,\varphi,\theta) & 0 \\ 0 & 0 & 0 & D(t,r,\varphi,\theta) \end{pmatrix} \tag{50}$$

with coordinates $t, r, \varphi, \theta$, computer algebra gives only 24 independent equations from (6). This means that 24 free parameters have to be determined arbitrarily to make the Γ connections unique. Since the covariant derivative is determined by the connections, this means that the covariant derivative of such a metric with torsion is not uniquely defined, in contrast to a torsion-free geometry. This seems to be a serious mathematical problem.

Adding anti-symmetry conditions for the last two indices of the Γ connection gives 4·6=24 additional conditions

$$\Gamma^{\rho}{}_{\mu\nu} = -\Gamma^{\rho}{}_{\nu\mu} \tag{51}$$

for all non-diagonal elements with $\mu \neq \nu$. Then the number of independent equations is increased to 60, but 4 dependent equations remain, leaving 4 free parameters undefined. The diagonal elements $\Gamma^{\rho}{}_{\mu\mu}$ are different from zero. If they are assumed to be zero, giving a fully anti-symmetric Γ connection, then the equation system is not solvable.

The above result remains valid if the metric has non-diagonal elements which depend on the coordinates of the row and column they are placed on, for example $g_{12}(t,r)$, $g_{13}(t,\varphi)$, etc., for example:

$$g = \begin{pmatrix} A(t,r,\varphi,\theta) & E(t,r) & F(t,\varphi) & G(t,\theta) \\ E(t,r) & B(t,r,\varphi,\theta) & 0 & 0 \\ F(t,\varphi) & 0 & C(t,r,\varphi,\theta) & 0 \\ G(t,\theta) & 0 & 0 & D(t,r,\varphi,\theta) \end{pmatrix}. \tag{52}$$

However, introducing an asymmetry or anti-symmetry in the metrical matrix prohibits a solution of the equation system, even without additional anti-symmetry conditions for the Γ connections.

The last investigated example is for the totally anti-symmetric torsion. Then the Γ connections are determined from equation (42) instead of (6). The result is again only 24 independent equations for metrics (50) and (52). If additional anti-symmetry is enforced (equation (51)), no solution is obtained. All results are summarized in Table 1. The equation system for standard anti-symmetric torsion behaves identical to that for fully anti-symmetric torsion if no additional anti-symmetry of the connections is assumed. When the latter is the case, there are no solutions for total anti-symmetric torsion.

| | Diagonal metric | Metric with special symm. non-diagonal elements $g_{\mu\nu}(x^\mu, x^\nu)$ | Metric with general symm. non-diagonal elements $g_{\mu\nu}(x^0, x^1, x^2, x^3)$ | Metric with general asymmetric non-diagonal elements |
|---|---|---|---|---|
| Basis eq.(6) | 24 | 24 | 24 | - |
| Basis eq.(6) + non-diagonal antisymm. cond. | 4 | 4 | - | - |
| Basis eq.(42) (fully antisymm. Torsion) | 24 | 24 | 24 | - |
| Basis eq.(42) (fully antisymm. Torsion) + non-diagonal antisymm. cond. | - | - | - | - |

Table 1: Number of independent equations from metric compatibility (- means equations not solvable).

## 7. Commutator Equation

The commutator of covariant derivatives leads to the well known commutator equation [2]

$$[D_\mu, D_\nu]V^\kappa = R^\kappa{}_{\lambda\mu\nu}V^\lambda - T^\lambda{}_{\mu\nu}D_\lambda V^\kappa \qquad (53)$$

where $V^\kappa$ is an arbitrary vector. The left hand side is anti-symmetric in μ, ν by definition. The Riemann tensor $R^\kappa{}_{\lambda\mu\nu}$ is defined anti-symmetrically in its last two indices and the same holds for torsion. Therefore both sides of the equation are not zero in general for $\mu \neq \nu$ but vanish for $\mu = \nu$. In the torsion tensor only the anti-symmetric part of the Γ connection is effective. If the connection is symmetric, torsion vanishes. However the Riemann tensor does not vanish since it is defined anti-symmetrically irrespective of the symmetry of the connection. Therefore, for a symmetric connection, the commutator equation simply reduces to

$$[D_\mu, D_\nu]V^\kappa = R^\kappa{}_{\lambda\mu\nu}V^\lambda \ . \qquad (54)$$

For $\mu = \nu$ both sides vanish again, but not for $\mu \neq \nu$.

## 8. Discussion and Conclusions

In earlier work where the symmetric connection of Einsteinian relativity (now obsolete) was used [2] it is convenient to start with a given metric and then compute the Christoffel symbols by equation (5). With knowledge of both the metric and the connections, all curvature elements of Riemann geometry (Riemann tensor, Ricci tensor, etc.) can be computed. The same could be done for Cartan geometry. Typically one would start with the tetrad which is the coordinate transformation between base manifold and tangent space. The metric then is computed by

$$g_{\mu\nu} = q^a{}_\mu q^b{}_\nu \, \eta_{ab} \qquad (55)$$

where $\eta_{ab}$ is the Minkowski metric. The Γ connection is computed from either (6) or (42) (in case of totally anti-symmetric torsion). By the Γ connection, torsion and curvature tensors can be determined as in case of Einsteinian relativity. However the Bianchi identity then contains torsion and has to be checked separately.

We have shown that the Christoffel connection or symbol has non-vanishing diagonal elements. Antisymmetry in the non-diagonal elements can be enforced but there is a problem of defining a unique covariant derivative for manifolds with torsion because the methods described in this paper do not lead to a unique solution for the Christoffel connections. This problem arises for torsion with anti-symmetry in the last two indices as well as in totally anti-symmetric torsion. We can compare this indeterminacy problem with Einsteinian relativity where it is an intrinsic feature. Since the Einstein field equations are generally covariant, a coordinate transformation of the form

$$x'^\mu = x'^\mu(x^0, x^1, x^2, x^3) \qquad (56)$$

can always be performed, letting the field equations unchanged. This means that the 10 independent elements of the metric have 4 free parameters, or 4 equations have to be defined additionally the make the metric – and the covariant derivative – unique. In Einsteinian theory this is a constraint on the Einstein tensor $G^{\mu\nu}$:

$$D_\nu G^{\mu\nu} = 0 \tag{57}$$

which is an invariance requirement consisting of 4 equations. Sometimes this is used as a “gauge condition” to find solutions of the field equation. In ECE theory, the coordinate transformation is already given by the tetrad and the metric in eq.(54) is uniquely defined. There is no option for free parameters in the metric. This cannot be possible because the tetrad is identical to the potential according to the first ECE axiom and this cannot be re-gauged.

If torsion is known, for example by solving the field equations, then the Christoffel connections can be computed from (14) in a direct and unique way. It does not seem possible to do it the "standard Riemann geometry way".

In future investigations a meaningful metric with torsion could be derived for examples of equation (55) and it could be seen if the same problems of uniqueness arise in this case.

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