CARTAN GEOMETRY OF VARIOUS GYROSCOPE MOTIONS.

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ABSTRACT

The acceleration in the spherical polar coordinate system is shown to be an example of a Cartan covariant derivative with well defined spin connection. The Lagrangian analysis of various gyroscope motions is given in terms of sets of simultaneous differential equations which are solved with Maxima code to give complete solutions. The underlying geometry is always Cartan geometry. The problems solved in this way are 1) the gyroscope in a gravitational field; 2) the gyroscope with point attached to a stand; 3) the theory of spherical orbits; 4) the general theory of a gyroscope in an external force field; 5) the general theory of the Milankowitch cycles.

Keywords: ECE theory, Cartan geometry, gyroscope motions, Milankovitch cycles.

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1. INTRODUCTION

In this series of three hundred and sixty nine papers and books to date {1 -12}, the ECE and ECE2 unified field theories are developed in terms of well known Cartan geometry {1 -12}. Therefore each papers an book is based on Cartan geometry, and the physics is developed as a variation of the central theme. In Section 2 the fundamental spherical polar coordinate system is shown to be an example of Cartan geometry with well defined spin connection. In general any curvilinear coordinate system is a variation on Cartan geometry, each coordinate system being defined by a spin connection. This new fundamental theorem is exemplified by applications to gyroscope motion of various kinds: 1) the gyroscope in a gravitational field; 2) the gyroscope attached to a stand; 3) the complete theory of spherical orbits; 4) the general theory of a gyroscope in an external force field; 5) the general theory of Milankovitch cycles.

The paper is a short synopsis of extensive and detailed calculations contained in the background notes accompanying UFT369 on www.aias.us. Note 369(1) describes the theory of a gyroscope in a gravitational field; Note 369(2) is the theory of a weightless gyroscop; Notes 369(3) is a preliminary theory of Milankovitch cycles; Notes 369(4) to 369(6) derive the spin connection of the spherical polar coordinate system and generalize the result to any curvilinear coordinate system; Note 369(7) gives a complete theory of spherical orbits; Note 369(8) gives the general theory of the gyroscope in an external field and Note 369(9) gives a general theory of Milankovitch cycles. These theories are variations on the theme of gyroscope motion, and each problem is solved with sets of simultaneous differential equations in the relevant Lagrange variables.

Section 2 is a short synopsis of the relevant Lagrangian analysis in each case. Section 3 solves the simultaneous differential equations and graphs the most important results.

2. VARIATION ON THE THEME OF GYROSCOPE MOTION

These variations are based on the relevant coordinate system, notably the spherical polar system. As in Note 369(4) it is shown to begin this section that spherical polar coordinate system is characterized by a well defined Cartan covariant derivative and spin connection matrix. The note begins by reviewing the derivation of the spin connection of the plane polar system, and proceeds to define the acceleration in the spherical polar system as the covariant Cartan derivative of the velocity. So all coordinate systems are examples of Cartan geometry. Consider the linear velocity {1 - 12} in the spherical polar system:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(1)$$

in standard notation {1 - 12}. The linear acceleration is given by:

$$\underline{a} = a_r\,\underline{e}_r + a_\theta\,\underline{e}_\theta + a_\phi\,\underline{e}_\phi \quad -(2)$$

where:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta \quad -(3)$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta \quad -(4)$$

$$a_\phi = 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi} + r\sin\theta\,\ddot{\phi}. \quad -(5)$$

The acceleration is defined by the following covariant Cartan derivative:

$$\frac{D}{Dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\sin\theta\,\dot{\phi}\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\sin\theta\,\dot{\phi}\end{bmatrix} + \begin{bmatrix}0 & -\dot{\theta} & -\sin\theta\,\dot{\phi}\\ \dot{\theta} & 0 & -\dot{\phi}\cos\theta\\ \sin\theta\,\dot{\phi} & \dot{\phi}\cos\theta & 0\end{bmatrix}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\sin\theta\,\dot{\phi}\end{bmatrix} \quad -(6)$$

in which the matrix on the right hand side is the spin connection, and is a rotation generator in three dimensions. The equivalent result in plane polar coordinates is:

$$\frac{D}{Dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix} + \begin{bmatrix}0 & -\dot{\theta}\\ \dot{\theta} & 0\end{bmatrix}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix} \quad -(7)$$

in which the two by two matrix is the spin connection, rotation generator in a plane.

The linear velocity in spherical polar coordinates is:

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(8)$$

and the dynamics of the unit vectors is described by:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta + \dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(9)$$

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r + \dot{\phi}\cos\theta\,\underline{e}_\phi \quad -(10)$$

$$\dot{\underline{e}}_\phi = -\dot{\phi}\sin\theta\,\underline{e}_r - \dot{\phi}\cos\theta\,\underline{e}_\theta \quad -(11)$$

It follows that Eq. (6) can be expressed as:

$$a_r = \frac{d\dot{r}}{dt} - \dot{\underline{e}}_r\cdot\underline{v} = \frac{dv_r}{dt} - \dot{\underline{e}}_r\cdot\underline{v} \quad -(12)$$

$$a_\theta = \frac{d}{dt}(r\dot{\theta}) - \dot{\underline{e}}_\theta\cdot\underline{v} = \frac{dv_\theta}{dt} - \dot{\underline{e}}_\theta\cdot\underline{v} \quad -(13)$$

$$a_\phi = \frac{d}{dt}(r\sin\theta\,\dot{\phi}) - \dot{\underline{e}}_\phi\cdot\underline{v} = \frac{dv_\phi}{dt} - \dot{\underline{e}}_\phi\cdot\underline{v} \quad -(14)$$

In the plane polar system the dynamics of the unit vectors can be summarized as:

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \quad -(15)$$

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r \quad -(16)$$

so it follows that Eq. (7) can be expressed as:

$$a_r = \frac{d\dot{r}}{dt} - \dot{\underline{e}}_r\cdot\underline{v} = \frac{dv_r}{dt} - \dot{\underline{e}}_r\cdot\underline{v} \quad -(17)$$

$$a_\theta = \frac{d}{dt}(r\dot{\theta}) - \dot{\underline{e}}_\theta\cdot\underline{v} = \frac{dv_\theta}{dt} - \dot{\underline{e}}_\theta\cdot\underline{v} \quad -(18)$$

It is seen that the set of equations (12) to (14) and (17) to (18) exhibit the same overall structure based on Cartan geometry. So the various gyroscope motions developed as follows are variations on the theme of Cartan geometry and general covariance in relativity theory.

The first example considers as in Note 369(7) the spherical orbit of a mass m about a mass M in a central force field:

$$\underline{F} = -\underline{\nabla} U = m\left(a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi\right) \quad -(19)$$

where the potential energy is:

$$U = -\frac{mMG}{r} \quad -(20)$$

Eq. (19) can be developed as:

$$\underline{F} = -\underline{\nabla} U = m\left(\frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v}\right) \quad -(21)$$

where $\underline{\omega}$ is the angular velocity in spherical polar coordinates and $\underline{v}$ the linear velocity. It follows that:

$$\underline{F} = m\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta\right)\underline{e}_r = -\frac{mMG}{r^2}\underline{e}_r \quad -(22)$$

so:

$$\ddot{r} - r\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) = -\frac{MG}{r^2} \quad -(23)$$

which is the Leibniz equation for a three dimensional orbit. For the central force field (21):

$$a_\theta \underline{e}_\theta + a_\phi \underline{e}_\phi = \underline{0} \quad -(24)$$

so

$$a_\theta^2 + a_\phi^2 = 0 \quad -(25)$$

where a_θ and a_ϕ are given by Eq. (2). The lagrangian for the spherical orbit is:

$$\mathcal{L} = \frac{1}{2} m v^2 - U \quad -(26)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) + \frac{mMG}{r}$$

and the three Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) \quad -(27)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) \quad -(28)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \quad -(29)$$

From Eqs. (26) and (27):

$$\ddot{r} = r \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) - \frac{MG}{r^2} \quad -(30)$$

which is Eq. (23), Q. E. D.

Eqs. (26) and (29) give the angular momentum L_ϕ as a constant of motion :

$$L_\phi = m r^2 \dot{\phi} \sin^2\theta \quad -(31)$$

and:

$$\frac{dL_\phi}{dt} = 0. \quad -(32)$$

Eqs. (26) and (28) give:

$$r \sin\theta \cos\theta \, \dot{\phi}^2 = r \ddot{\theta} + 2 \dot{\theta} \dot{r} \quad -(33)$$

so as shown in Note 369(7) the complete solution for any spherical orbit is given by solving the following three simultaneous differential equations:

$$\ddot{r} = r\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - \frac{mG}{r^2} \quad -(34)$$

$$r \sin\theta \cos\theta \, \dot{\phi}^2 = r\ddot{\theta} + 2\dot{\theta}\dot{r} \quad -(35)$$

$$\frac{2\dot{r}L_\phi}{mr^2} + 2r\dot{\phi}\dot{\theta}\cos\theta\sin\theta + r\ddot{\phi}\sin^2\theta = 0. \quad -(36)$$

This is carried out in Section 3 with Maxima code. In general, all orbits are spherical orbits.

Notes 369(1) and 369(2) give complete details of the motion of a gyroscope in a gravitational field, using the Lagrangian method. Readers are referred to these notes for details. They develop the results of UFT368. An important result is that the condition for weightlessness of the gyro is:

$$\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t) = \frac{mG}{r_e^2} \quad -(37)$$

and this theory supports the reproducible and repeatable result of Laithwaite described in UFT368. Under certain well defined conditions the gyroscope can appear to be weightless.

In Note 369(8) a general theory is given of a gyroscope in an external force field, using the lagrangian:

$$\mathcal{L} = \frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - U. \quad -(38)$$

The translational kinetic energy is given by:

$$T(\text{trans}) = \frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} \quad -(39)$$

where $\underline{r}$ is the position in the lab frame of the centre of mass of the gyroscope of mass m. The rotational kinetic energy is defined in the frame (1, 2, 3) of the principal moments of inertia I_1, I_2, and I_3 of the gyroscope. Here ω_1, ω_2 and ω_3 are the angular velocities in axes 1, 2, and 3. In general {1 - 12} the potential energy is defined by:

$$\int_1^2 \underline{F} \cdot \underline{dr} = U_1 - U_2 \quad -(40)$$

in terms of the work done by the force $\underline{F}$ to transport the centre of mass of the gyroscope from point 1 to point 2. A solution of Eq. (40) is:

$$\underline{F} = -\underline{\nabla} U \quad -(41)$$

corresponding to the external torque:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(42)$$

applied to the gyroscope. The potential energy, a scalar, is the same in the lab frame (X, Y, Z) and in the frame (1, 2, 3).

In terms of the Euler angles θ, ϕ, χ {1 - 12}, the angular velocities are:

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin\theta \sin\chi + \dot{\theta} \cos\phi \quad -(43)$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\chi}_2 = \dot{\phi} \sin\theta \cos\chi - \dot{\theta} \sin\phi \quad -(44)$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\chi}_3 = \dot{\phi} \cos\theta + \dot{\phi} \quad -(45)$$

and for a symmetric top:

$$T(\text{rot}) = \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 \quad -(46)$$

as in UFT368, where:

$$I_{12} = I_1 = I_2 \quad -(47)$$

The general lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + T(\text{rot}) - (U_1 - U_2) \quad -(48)$$
$$= \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + T(\text{rot}) - \int_1^2 \underline{F} \cdot d\underline{r}$$

in which any kind of external force $\underline{F}$ can be defined in the laboratory frame (X, Y, Z).

UFT368 solves the problem of a symmetric top gyroscope whose point is fixed, and whose lagrangian is:

$$\mathcal{L} = T(\text{rot}) - mgh\cos\theta \quad -(49)$$

in the notation of UFT368. The gyroscope with a fixed point cannot translate, so:

$$\dot{\underline{r}} = \underline{0}. \quad -(50)$$

If the point of the gyro is allowed to move, and an extrenal force applied, te lagrangian (49) generalizes to:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + T(\text{rot}) - mgh\cos\theta - \int_1^2 \underline{F} \cdot d\underline{r} \quad -(51)$$

in which

$$\dot{\underline{r}} \neq \underline{0}. \quad -(52)$$

Writing:

$$U := U_1 - U_2 \quad -(53)$$

the lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + T(\text{rot}) - mgh\cos\theta - U \quad -(54)$$

The lagrange variables are $\underline{r}, \theta, \phi, \psi$ and the dynamics of the gyroscope are described by the four Euler Lagrange equations:

$$\underline{\nabla} \mathcal{L} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}}\right) \quad -(55)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(56)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \quad -(57)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right). \quad -(58)$$

For a potential energy that depends only on r

$$U = U(r) \quad -(59)$$

these Euler Lagrange equations give (UFT368):

$$\dot{\phi} = \frac{L_\phi - L_\psi \cos\theta}{I_{12} \sin^2\theta} \quad -(60)$$

$$\dot{\psi} = \frac{1}{I_3}\left(L_\psi - I_3 \dot{\phi} \cos\theta\right) \quad -(61)$$

$$\ddot{\theta} = \frac{\sin\theta}{I_{12}}\left(\dot{\phi}^2 \cos\theta \left(I_{12} - I_3\right) - I_3 \dot{\phi}\dot{\psi} + mgh\right) \quad -(62)$$

and

$$m \frac{d\underline{r}}{dt} = -\underline{\nabla} U(r). \quad -(63)$$

For a potential energy of type (59) the translational motion of the centre of mass of the gyroscope is independent of its own rotational motion (nutations and precessions). However in the general case:

$$U = U(r, \theta, \phi, \chi) \quad (64)$$

the rotational and translational motions are inter related.

Finally, Note 369(9) considers the general case of the Milankowitch cycles, which are thought of as being due to the nutations and precessions of the asymmetric top gyroscope of mass m in the gravitational field of the sun of mass M. The distance between the sun and the centre of mass of the gyroscope is $\underline{r}$ in the laboratory frame (X, Y, Z). The distance between the centre of mass and a point in the gyroscope is $\underline{r}_1$ in the same laboratory frame. So the lagrangian is:

$$\mathcal{L} = \frac{1}{2} m (\dot{\underline{r}} + \dot{\underline{r}}_1) \cdot (\dot{\underline{r}} + \dot{\underline{r}}_1) - U \quad (65)$$

where U is the gravitational potential:

$$U = -\frac{mMG}{r}. \quad (66)$$

The position of the centre of mass is:

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k}. \quad (67)$$

The position of $\underline{r}_1$ is defined in the (1, 2, 3) frame of the principal moments of inertia:

$$\underline{r}_1 = r_{11}\underline{e}_1 + r_{12}\underline{e}_2 + r_{13}\underline{e}_3 \quad (68)$$

in which r_{11}, r_{12}, and r_{13} are constants defined by the shape of the gyroscope and related to the principal moments of inertia.

So the combined motion is nutation and precession of the gyroscope superimposed on orbital motion.

The transformation from (1, 2, 3) to (X, Y, Z) is given by {1 -12}

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(69)$$

in terms of the Euler angles. Therefore:

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} \quad -(70)$$

where

$$\begin{aligned} A &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\ B &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\ C &= \sin\theta\cos\phi \\ D &= \cos\psi\cos\phi + \cos\theta\cos\phi\sin\psi \\ E &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\ F &= -\sin\theta\cos\phi \\ G &= \sin\psi\cos\theta \\ H &= \cos\psi\sin\theta \\ I &= \cos\theta . \end{aligned} \quad -(71)$$

It follows that:

$$\begin{aligned} \underline{r}_1 &= r_{11}(A\underline{i} + B\underline{j} + C\underline{k}) + r_{12}(D\underline{i} + E\underline{j} + F\underline{k}) + r_{13}(G\underline{i} + H\underline{j} + I\underline{k}) \\ &= (r_{11}A + r_{12}D + r_{13}G)\underline{i} + (r_{11}B + r_{12}E + r_{13}H)\underline{j} + (r_{11}C + r_{12}F + r_{13}I)\underline{k} \\ &= X_1\underline{i} + Y_1\underline{j} + Z_1\underline{k} \quad -(72) \end{aligned}$$

The lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}m\left((\dot{X}+\dot{X}_1)^2 + (\dot{Y}+\dot{Y}_1)^2 + (\dot{Z}+\dot{Z}_1)^2\right) + \frac{mMG}{r} \quad -(73)$$

If the orbit is considered to be planar, then:

$$\begin{aligned} X &= r\cos\theta_1 \\ Y &= r\sin\theta_1 \\ Z &= 0 \end{aligned} \quad -(74)$$

The dynamics are therefore given by five Euler Lagrange equations in five Lagrange variables as follows

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \\ \frac{\partial \mathcal{L}}{\partial \theta_1} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) \\ \frac{\partial \mathcal{L}}{\partial \theta} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \\ \frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \\ \frac{\partial \mathcal{L}}{\partial \alpha} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) \end{aligned} \quad -(75)$$

These must be solved simultaneously. Note that:

$$\begin{aligned} \dot{X} &= \dot{r}\cos\theta_1 - r\dot{\theta}_1\sin\theta_1 \\ \dot{Y} &= \dot{r}\sin\theta_1 + r\dot{\theta}_1\cos\theta_1 \\ \dot{Z} &= 0 \end{aligned} \quad -(76)$$

and:

$$\dot{X}_1 = r_{11}\frac{dA}{dt} + r_{12}\frac{dD}{dt} + r_{13}\frac{dG}{dt}$$

$$\dot{Y}_1 = r_{11}\frac{dB}{dt} + r_{12}\frac{dE}{dt} + r_{13}\frac{dH}{dt} \qquad -(77)$$

$$\dot{Z}_1 = r_{11}\frac{dC}{dt} + r_{12}\frac{dF}{dt} + r_{13}\frac{dI}{dt}$$

These can be evaluated by computer algebra to eliminate human error. The lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}m\left(\left(\dot{r}\cos\theta_1 - r\dot{\theta}_1\sin\theta_1 + \dot{X}_1\right)^2 + \left(\dot{r}\sin\theta_1 + r\dot{\theta}_1\cos\theta_1 + \dot{Y}_1\right)^2 + \dot{Z}_1^2\right) + \frac{mMG}{r} \qquad -(78)$$

It is seen that the Milankowitch cycles are the result of intricate dynamics which are all inter related. The solution of this problem is discussed in Section 3 using numerical methods.

3. NUMERICAL SOLUTIONS AND GRAPHICS OF SELECTED RESULTS.

Section by Dr. Horst Eckardt

Cartan geometry in various gyroscope motions

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3 Numerical solutions and graphics of selected results

We will consider three special cases of gyroscopic motion in this section.

3.1 Motion of a free falling gyroscope

We compute the motion of a symmetric top with one point fixed, where the fixed point is freely moving in the Z direction. This is a free falling gyroscope. The model of the gyro fixed having Euler angles θ, ϕ, ψ is extended by a coordinate R representing the Z motion, see Fig. 1. Then the rotational part of the Lagrangian is according to Eq. (46):

$$T_{rot} = \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin(\theta)^2 + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos(\theta) + \dot{\psi} \right)^2 . \tag{79}$$

The translational part has to be extended by an $\dot{R}$ term, representing the velocity in Z direction:

$$T_{trans} = \frac{m}{2} \left(\dot{R} + h \dot{\theta} \sin(\theta) \right)^2 . \tag{80}$$

Correspondingly the potential energy is

$$U = mg \left(h \cos(\theta) + R \right) \tag{81}$$

with gravitational acceleration g. The Lagrangian is

$$\mathscr{L} = T_{rot} + T_{trans} - U. \tag{82}$$

The four Lagrange equations consist of three equations for θ, ϕ, ψ as before plus an additinal one for the R coordinate:

$$\ddot{R} = h \ddot{\theta} \sin(\theta) + h \dot{\theta}^2 \cos(\theta) - g. \tag{83}$$

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There is a coupling between the second derivatives of R and θ, therefore this is not the canonical form. Resolving the Lagrange equation system of four unknowns $\ddot{\theta}, \ddot{\phi}, \ddot{\psi}, \ddot{R}$ then gives the canonical equations

$$\ddot{\theta} = \frac{\left((I_{12} - I_3)\, \dot{\phi}^2\, \cos(\theta) - I_3\, \dot{\phi}\, \dot{\psi}\right)\, \sin(\theta)}{I_{12}}, \tag{84}$$

$$\ddot{\phi} = -\frac{\left((2I_{12} - I_3)\, \dot{\phi}\, \cos(\theta) - I_3\, \dot{\psi}\right)\, \dot{\theta}}{I_{12}\, \sin(\theta)}, \tag{85}$$

$$\ddot{\psi} = \frac{\left((I_{12} - I_3)\, \dot{\phi} \cos(\theta)^2 - I_3\, \dot{\psi}\, \cos(\theta) + I_{12}\, \dot{\phi}\right)\, \dot{\theta}}{I_{12}\, \sin(\theta)}, \tag{86}$$

$$\ddot{R} = h\, \cos(\theta)\, \dot{\theta}^2 + \frac{\left((I_{12} - I_3)\, h\, \dot{\phi}^2\, \cos(\theta) - I_3 h\, \dot{\phi}\, \dot{\psi}\right)\, \sin(\theta)^2}{I_{12}} - g. \tag{87}$$

The numerical solution of Eqs. (84-87) is presented in Figs. 2-4 for a sutable set of initial conditions. From Fig. 2 can be seen that the angles of nutation and precession (θ and ϕ) oscillate with phase shift while the angel of rotation (ψ) increases roughly linearly, i.e. the rotation speed is nearly constant but modulated by the other angle positions. There is an impact on the rotation of the rigid body around its body axis. This effect cannot occur when only two angles of polar coordination system (θ, ϕ) are used, then the degree of freedom is lowered by one and essential information is missing. The gyroscope is moving in free fall in negative Z direction. This can be seen from the free fall parabola of R (Fig. 3). The velocity would be linear in such a case but is modulated by the angular precession. The space curve of the centre of mass (Fig. 4) shows an elliptic helix with variable pitch due to the acceleration in $-Z$ direction.

3.2 Explanation of Laithwaite experiment

In the Laithwaite experiment a spinning top is lifted in a way that during lift-off the axis is moved in a manner that the force for lifting obviously is small compared to the weight of the spinning top. It may be that the initial conditions of nutation are modified so that the gravitational force is counteracted for a moment. This means that

$$\ddot{R} = 0 \tag{88}$$

for this moment. Inserting this in Eq. (87) and assuming $I_{12} = I_3$ for simplicity, leads to the condition

$$g = h\, \dot{\theta}^2\, \cos(\theta) - h\, \dot{\phi}\dot{\psi} \sin(\theta)^2. \tag{89}$$

The simulation results show that such a condition can be met by a strong negative initial value of $\dot{\theta}$ giving the spinning top a kick. Then the position of the fixed point (held by hand) overshoots the initial position as shown in Fig. 5. The vertical velocity $v = \dot{R}$ is positive and oscillates relatively strongly during the later motion. The condition of weightlessness, Eq. (89), is graphed in Fig. 6. The right hand side and left hand side of the equation are plotted, and the spinning top is weightless at the intersection points of both curves. The details depend on the parameters chosen. Our calculation is a model calculation without using parameters of the real system because this requires a considerable recherche effort. We can show that the Laithwaite experiment is possible in principle on the basis of classical dynamics.

3.3 External torque

An external torque can be introduced by a generalized force into the Lagrange mechanism. Since we ware working with potentials here, we define a potential giving a constant torque T_{q0} in Z direction (for the angle ϕ) by

$$T_q = -\frac{\partial U_q}{\partial \phi} \tag{90}$$

with

$$U_q = -T_{q0}\,\phi \tag{91}$$

and adding this to the potential energy:

$$U = mg\,(h\cos(\theta) + R) + U_q. \tag{92}$$

This term - if chosen not too small - has an enormous impact on the motion of the gyroscope. The results can be very exotic in dependence of the value of T_{q0} and the initial conditions. Fig. 7 shows the three angular frequencies. There is an initial phase where the ϕ rotation remains constant although an external force is being applied. After this phase, $\dot{\phi}$ increases linearly in average due to the torque as expected. The corresponding angular trajectories are graphed in Fig. 8. The angle θ shows a nutation. Interestingly, the self-rotation of the gyroscope changes direction after the initial phase, standing still for a short moment (crossing of zero axis). The vertical velocity (Fig. 9) shows strong oscillations which are even detectable in the linear motion R. The initial phase is clearly discernible from the rest by inspecting the space curve of the centre of mass (Fig. 10). After some irregular initial motions the ϕ rotation dominates.

Other, more complicated effects emerge when T_q is made periodic in time, for example

$$T_q = T_{q0}\cos(\omega t) \tag{93}$$

with a time frequency ω. Then new effects like heterodynes in angular velocities can appear, see Fig. 11 as an example. In this case there is no continuous rotation in ϕ direction. By suitable initial conditions, it is even possible to stop all rotations.

What could not be confirmed is the lifting effect experimentally investigated by Shipov. By applying a torque in ϕ (i.e. around the Z direction) a gyroscope should lose weight. This would be an increase of linear momentum against gravitational force. Although momenta can be exchanged between all kinds of motion, a Lagrangian formalism conserves total momentum. This could be circumvented by applying an external torque, but it seems that several kinds of torque must be switched on and off in a complicated way to give a resulting linear motion against the gravitational force. A simple ϕ torque seems not to reproduce such an effect.

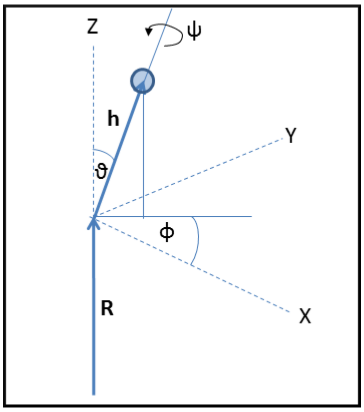

Figure 1: Geometry of a free falling gyroscope with one point fixed.

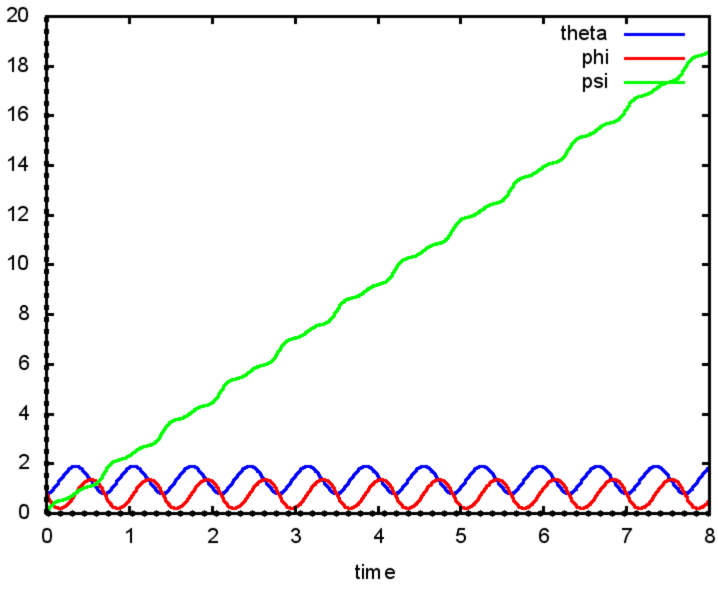

Figure 2: Trajectories $\theta(t)$, $\phi(t)$, $\psi(t)$ for a free falling gyroscope.

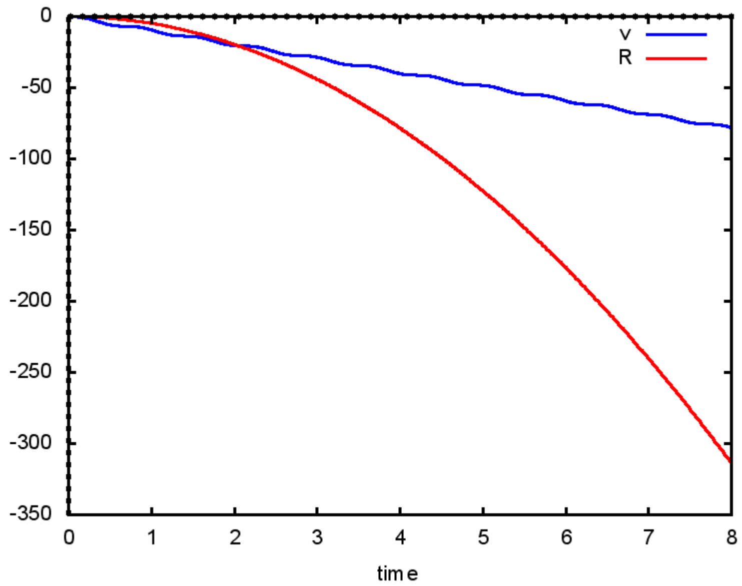

Figure 3: Vertical velocity $v(t)$ and height $R(t)$ for a free falling gyroscope.

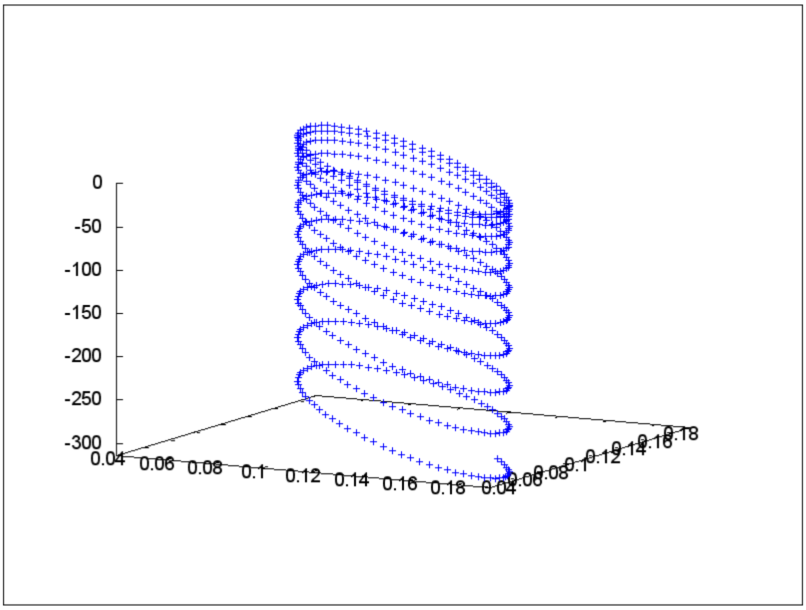

Figure 4: Space curve for a free falling gyroscope.

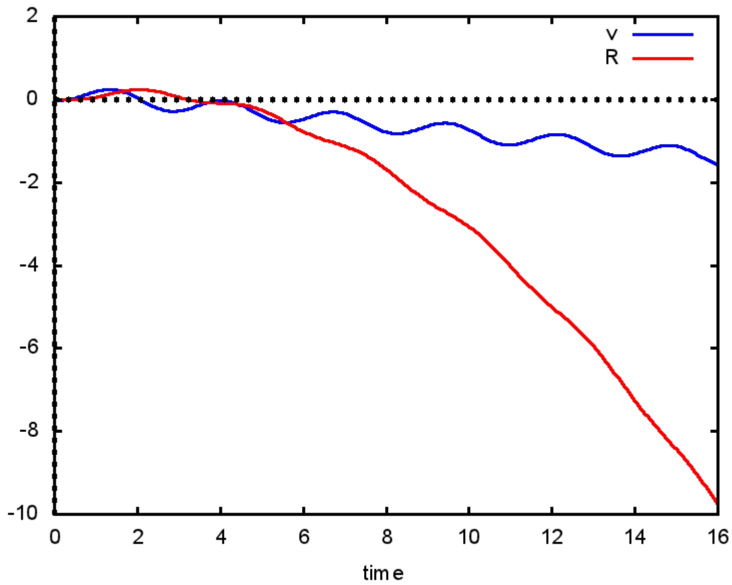

Figure 5: Example values $v(t)$ and $R(t)$ for Laithwaite experiment.

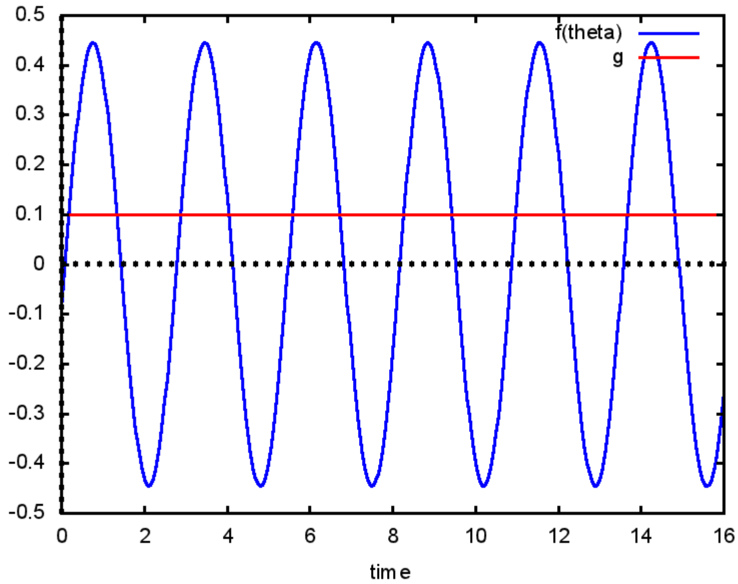

Figure 6: Both sides of Eq. (89) demonstrating weightless points.

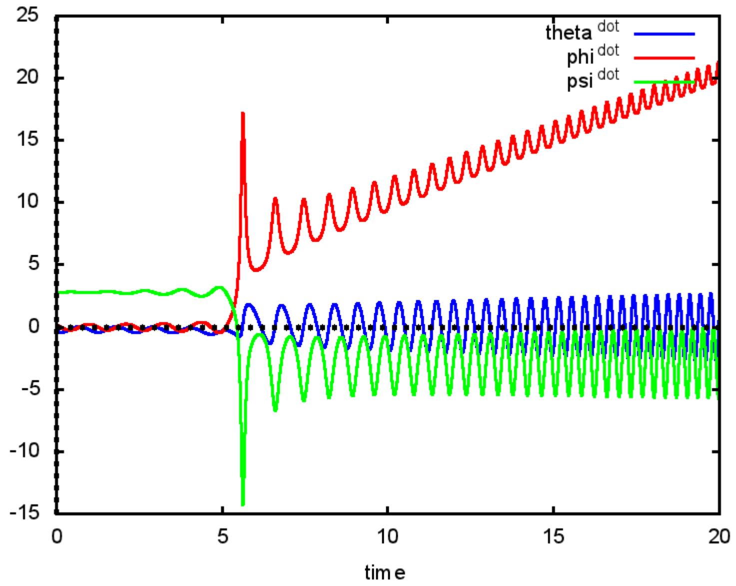

Figure 7: Trajectories $\dot{\theta}(t)$, $\dot{\phi}(t)$, $\dot{\psi}(t)$ for an external ϕ torque.

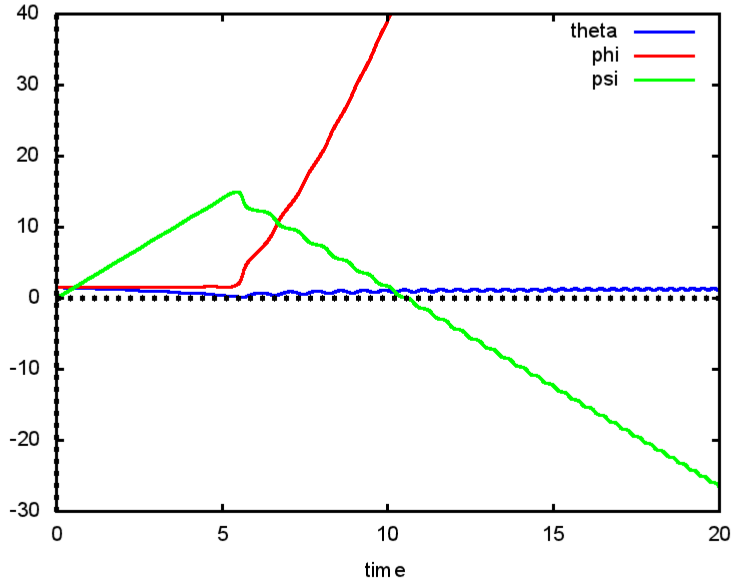

Figure 8: Trajectories $\theta(t)$, $\phi(t)$, $\psi(t)$ for an external ϕ torque.

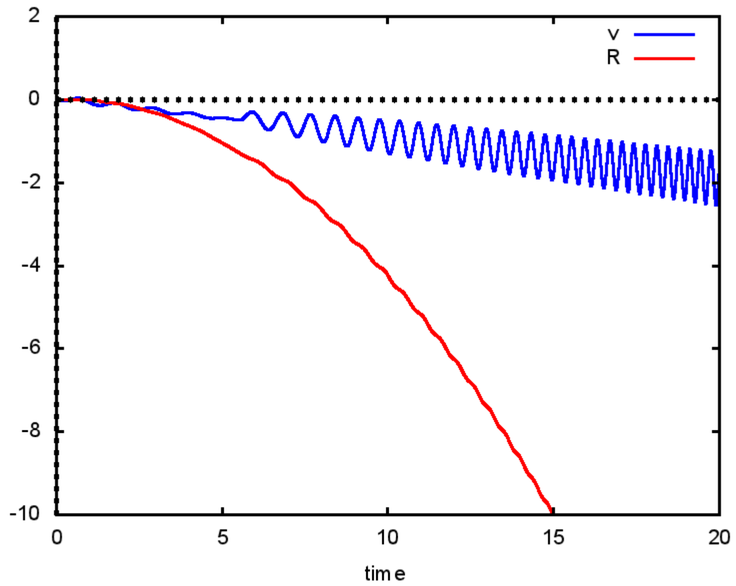

Figure 9: Trajectories $v(t)$, $R(t)$ for an external ϕ torque.

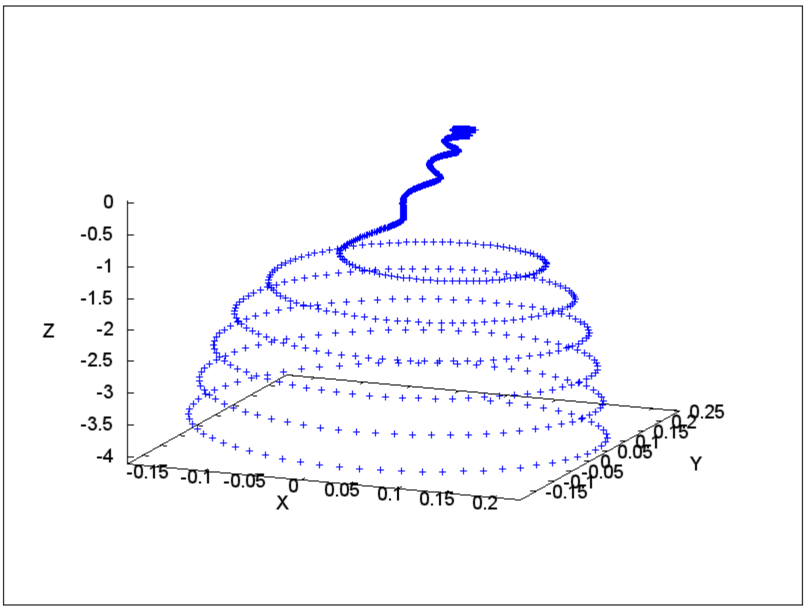

Figure 10: Space curve for an external ϕ torque.

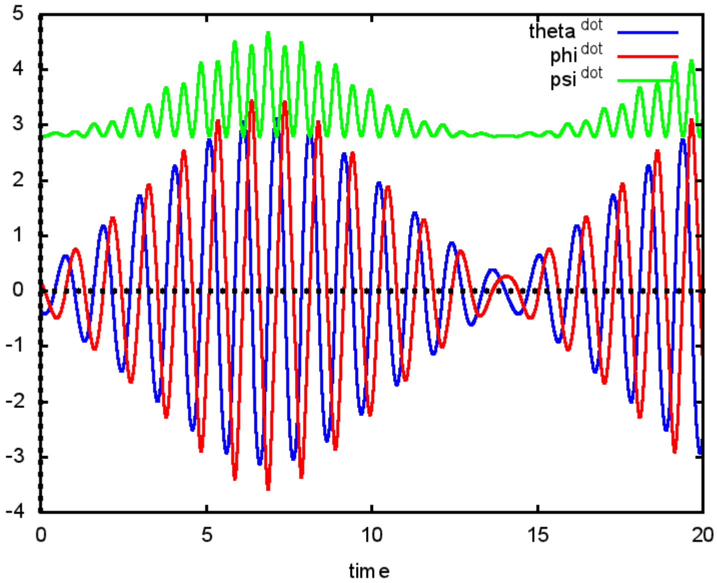

Figure 11: Angular velocities for an external, time-varying ϕ torque.

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