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Generalized Cartan Bianchi Identity and a New Theorem of the Cartan Torsion

by

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Abstract

It is shown that the commutator of covariant derivatives acting on the four vector produces in general two tensors which are inter-related by a generalized form of the well known Bianchi identity of differential geometry as developed by Cartan. Examples of this theorem are the original Cartan/Bianchi identity itself, its Hodge dual identity, and its derivative identity. By using the well known rule for covariant derivative of a rank three tensor a new cyclic identity of the Cartan torsion is proven. In general these two tensors always exist in four dimensional space-time, and their existence is a direct consequence of the definition of covariant derivative in four dimensions. Therefore any self-consistent theory of relativity must be based on this self-consistent geometry.

Keywords: Generalized Cartan/Bianchi identity, new theorem of the Cartan torsion, theory of relativity.

16.1 Introduction

It is well known that the differential geometry of Cartan [1] is developed in terms of two structure equations and the Bianchi identity. The structure equations define the Cartan torsion in terms of the tetrad, and the Cartan curvature in terms of the spin connection. The Cartan/Bianchi identity relates the torsion and curvature. Recently the Einstein Cartan Evans (ECE) theory has been developed into a self-consistent and generally covariant unified field theory based directly on these structure equations and Bianchi identity [2–10]. The ECE theory produces all the equations of physics from the Cartan geometry, and unifies relativity and wave mechanics using the tetrad postulate. The latter is the fundamental theorem that links Cartan to Riemann geometry. In this paper it is proven that the Cartan/Bianchi identity can be generalized to produce two tensors whose existence depends only on the fundamental definition of the covariant derivative itself in four dimensions. This theorem is proven in Section 16.2. Examples of the theorem are the Cartan/Bianchi identity itself, its Hodge dual identity, and its derivative identity. This self-consistent theorem of geometry is fundamental to any valid theory of relativity in physics, or natural philosophy. Any theory that arbitrarily assumes that one tensor is zero is self-inconsistent geometrically and cannot produce a correct description of physics. An example of a self-inconsistent theory is the Einstein Hilbert field theory of gravitation, which is based on an incorrect geometry, so cannot produce correct physics. In Section 16.3 a new theorem of the Cartan torsion is proven from the general theorem of Section 16.2.

16.2 Generalized Cartan/Bianchi Identity

Define any two anti-symmetric tensors by:

$$[D_\mu,\ D_\nu]V^\kappa := A^\kappa{}_{\sigma\mu\nu}V^\sigma - B^\lambda_{\mu\nu}D_\lambda V^\kappa \tag{16.1}$$

i.e. by the action of the commutator of covariant derivatives:

$$[D_\mu,\ D_\nu] = -[D_\nu,\ D_\mu] \tag{16.2}$$

on the four vector V^κ. The two tensors are always related by the generalized Cartan/Bianchi identity:

$$D \wedge B^a = A^a{}_b \wedge q^b \tag{16.3}$$

where in Eq. (16.3) the standard notation of differential geometry [1–10] has been used.

The proof of this theorem relies only on the fundamental definition of the covariant derivative:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda \tag{16.4}$$

where Γ is the connection. Eq. (16.4) implies that the structure of the two tensors must be:

$$A^\kappa{}_{\sigma\mu\nu} := \partial_\mu\Gamma^\kappa_{\nu\sigma} - \partial_\nu\Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{16.5}$$

and

$$B^{\kappa}_{\mu\nu} := \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} \tag{16.6}$$

and that the two tensors must be anti-symmetric as follows:

$$A^{\kappa}{}_{\sigma\mu\nu} = -A^{\kappa}{}_{\sigma\nu\mu}, \tag{16.7}$$

$$B^{\kappa}_{\mu\nu} = -B^{\kappa}{}_{\nu\mu}. \tag{16.8}$$

Eqs. (16.5) to (16.8) follow directly from Eq. (16.1) without any assumption other than Eq. (16.4). Therefore any two tensors with the structures (16.5) and (16.6) will obey the generalized Cartan/Bianchi identity (16.3). Both tensors are generated directly from the commutator (16.2) and it is incorrect to assert that one tensor must vanish. In general both tensors must be non-zero and both must exist in the same space-time. Given the structure (16.5), the structure (16.6) follows. This result is true for all metrics and all connections, and is true irrespective of any postulate of metric compatibility [1–10]. The result depends only on the anti-symmetry (16.2) of the commutator of covariant derivatives. The latter is basic to field theory as is well known [11, 12].

It is necessary to prove that if the anti-symmetric tensors (16.5) and (16.6) exist from the commutator (16.2), then they always obey Eq. (16.3). In tensor notation, Eq. (16.3) is:

$$D_\mu B^a_{\nu\rho} + D_\rho B^a_{\mu\nu} + D_\nu B^a_{\rho\mu} = A^a_{\mu\nu\rho} + A^a_{\rho\mu\nu} + A^a_{\nu\rho\mu}. \tag{16.9}$$

The covariant derivatives on the left hand side of this equation are expanded [1–10] with the spin connection as follows:

$$\begin{aligned}\partial_\mu B^a_{\nu\rho} + \omega^a_{\mu b} B^b_{\nu\rho} + \partial_\rho B^a_{\mu\nu} + \omega^a_{\rho b} B^b_{\mu\nu} + \partial_\nu B^a_{\rho\mu} + \omega^a_{\nu b} B^b_{\rho\mu} \\ = (A^\lambda_{\mu\nu\rho} + A^\lambda_{\rho\mu\nu} + A^\lambda_{\nu\rho\mu}) q^a_\lambda\end{aligned} \tag{16.10}$$

where:

$$B^a_{\nu\rho} = (\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu}) q^a_\lambda \quad \text{etc.}, \tag{16.11}$$

$$B^b_{\nu\rho} = (\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu}) q^b_\lambda \quad \text{etc.} \tag{16.12}$$

The Cartan tetrad is defined by the tetrad postulate [1–10]:

$$D_\mu q^a_\sigma = \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Gamma^\lambda_{\mu\sigma} q^a_\lambda = 0 \tag{16.13}$$

Therefore the tetrad postulate links the spin connection and gamma connection:

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma = \Gamma^\lambda_{\mu\sigma} q^a_\lambda. \tag{16.14}$$

Using the Leibnitz Theorem:

$$\partial_\mu B^a_{\nu\rho} = (\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}) q^a_\lambda + (\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu}) \partial_\mu q^a_\lambda \quad \text{etc.,} \tag{16.15}$$

so Eq. (16.10) becomes:

$$\begin{aligned} &(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}) q^a_\lambda + (\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu})(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda) + \ldots \\ &\quad = (A^\lambda_{\mu\nu\rho} + A^\lambda_{\rho\mu\nu} + A^\lambda_{\nu\rho\mu}) q^a_\lambda. \end{aligned} \tag{16.16}$$

Now re-label the summation indices in the second term on the left hand side as follows:

$$\lambda \rightarrow \sigma. \tag{16.17}$$

These are the repeated indices or dummy indices of the covariant - contravariant notation, and are summed over by definition. They can therefore take any label and after this re-labeling Eq. (16.16) becomes:

$$\begin{aligned} &(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}) q^a_\lambda + (\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu})(\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma) + \ldots \\ &\quad = (A^\lambda_{\mu\nu\rho} + A^\lambda_{\rho\mu\nu} + A^\lambda_{\nu\rho\mu}) q^a_\lambda. \end{aligned} \tag{16.18}$$

Finally use the tetrad postulate (16.14) to obtain:

$$\begin{aligned} A^\lambda_{\mu\nu\rho} + A^\lambda_{\rho\mu\nu} + A^\lambda_{\nu\rho\mu} &= \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\mu\sigma}(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}) \\ &+ \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\rho\sigma}(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}) \\ &+ \partial_\mu \Gamma^\lambda_{\rho\mu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\nu\sigma}(\Gamma^\sigma_{\rho\mu} - \Gamma^\sigma_{\mu\rho}) \end{aligned} \tag{16.19}$$

Re-arrange this cyclic sum as follows:

$$\begin{aligned} &A^\lambda_{\rho\mu\nu} + A^\lambda_{\mu\nu\rho} + A^\lambda_{\nu\rho\mu} \\ &\quad = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho} \\ &\quad\quad + \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\rho\mu} - \Gamma^\lambda_{\rho\sigma} \Gamma^\sigma_{\nu\mu} \\ &\quad\quad + \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\rho\sigma} \Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\rho\nu}. \end{aligned} \tag{16.20}$$

This is precisely the cyclic sum of three definitions (16.5). In order to obtain this result the definition (16.6) must be used. So the two tensors $A^{\kappa}_{\sigma\mu\nu}$ and $B^{\kappa}_{\mu\nu}$ are related by Eq. (16.3), Q.E.D.

Note that Eq. (16.20) is the cyclic sum of three tensors (16.5) on one side, and the cyclic sum of the definitions of these same tensors on the other side. So Eq. (16.20) and therefore Eq. (16.3) are exact identities which are obeyed by ANY two tensors with the structures (16.5) and (16.6). As seen from Eq. (16.1) these two tensors always exist in four dimensions. Multiply both sides of Eq. (16.9) by the tetrad q^{κ}_{a}:

$$(D_\mu B^a_{\nu\rho} + D_\rho B^a_{\mu\nu} + D_\nu B^a_{\rho\mu}) q^\kappa_a = (A^a_{\mu\nu\rho} + A^a_{\rho\mu\nu} + A^a_{\nu\rho\mu}) q^\kappa_a \tag{16.21}$$

to find that the following is a particular solution of Eq. (16.3):

$$D_\mu B^\kappa_{\nu\rho} + D_\rho B^\kappa_{\mu\nu} + D_\nu B^\kappa_{\rho\mu} = A^\kappa_{\mu\nu\rho} + A^\kappa_{\rho\mu\nu} + A^\kappa_{\nu\rho\mu}. \tag{16.22}$$

Eq. (16.22) is expressed in the base manifold (a four dimensional space-time) and eliminates the tangent index a of Cartan geometry [1–10].

The Cartan/Bianchi identity is recovered if:

$$A^\kappa_{\mu\nu\rho} = R^\kappa_{\mu\nu\rho}, \quad B^\kappa_{\nu\rho} = T^\kappa_{\nu\rho} \tag{16.23}$$

where $R^\kappa_{\mu\nu\rho}$ is the curvature tensor and $T^\kappa_{\nu\rho}$ the torsion tensor. Another important example of Eq. (16.22) is found by using the Hodge dual of the commutator operator as follows:

$$[D^\mu, D^\nu]_{\mathrm{HD}} = \frac{1}{2} \parallel g \parallel^{\frac{1}{2}} \epsilon^{\mu\nu\alpha\beta} [D_\alpha, D_\beta] \tag{16.24}$$

Here $\|g\|^{\frac{1}{2}}$ is the square root of the determinant of the metric [1–10], and $\epsilon^{\mu\nu\alpha\beta}$ is the totally anti-symmetric unit tensor in four dimensions. The latter is defined in the same way [1–10] as in Minkowski space-time. From Eqs. (16.1) and (16.24):

$$[D_\mu, D_\nu]_{\mathrm{HD}} V^\kappa = \widetilde{A}^\kappa{}_{\sigma\mu\nu} V^\sigma - \widetilde{B}^\lambda_{\mu\nu} D_\lambda V^\kappa \tag{16.25}$$

where:

$$\widetilde{A}^\kappa{}_{\sigma\mu\nu} := (\partial_\mu \Gamma^\kappa_{\nu\sigma} - \partial_\nu \Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\kappa_{\nu\lambda} \Gamma^\lambda_{\mu\sigma})_{\mathrm{HD}}. \tag{16.26}$$

and

$$\widetilde{B}^\kappa_{\mu\nu} := (\Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu})_{\mathrm{HD}}. \tag{16.27}$$

The Hodge dual tensors (16.26) and (16.27) are examples of the tensors $A^{\kappa}{}_{\sigma\mu\nu}$ and $B^{\kappa}_{\mu\nu}$ and so are related by:

$$D \wedge \widetilde{B}^a = \widetilde{A}^a{}_b \wedge q^b. \tag{16.28}$$

For example:

$$\widetilde{A}^{\kappa}{}_{\sigma}{}^{01} = \partial_2 \Gamma^{\kappa}_{3\sigma} - \partial_3 \Gamma^{\kappa}_{2\sigma} + \Gamma^{\kappa}_{2\lambda}\Gamma^{\lambda}_{3\sigma} - \Gamma^{\kappa}_{3\lambda}\Gamma^{\lambda}_{2\sigma} \tag{16.29}$$

and:

$$\widetilde{B}^{\kappa 01} = \Gamma^{\kappa}_{23} - \Gamma^{\kappa}_{32} \tag{16.30}$$

and these are related to each other in the same way as Eqs. (16.5) and (16.6), using the same connections.

Therefore we have proven rigorously that if:

$$D \wedge B^a = A^a{}_b \wedge q^b \tag{16.31}$$

then:

$$D \wedge \widetilde{B}^a = \widetilde{A}^a{}_b \wedge q^b. \tag{16.32}$$

Eq. (16.32) is referred to as the Hodge dual of the Cartan/Bianchi identity and was first inferred during the development of ECE theory [2–10]. It is clear that there also exists the derivative identity:

$$D \wedge (D \wedge B^a) := D \wedge (A^a{}_b \wedge q^b) \tag{16.33}$$

which is the correct form of the so called "second Bianchi identity" of the standard model.

In paper 93 of the ECE theory (www.aias.us) it was shown using computer algebra that curvature tensors of the type $R^{\kappa}{}_{\mu}{}^{\mu\nu}$ are non-zero in general if calculated from exact solutions of the Einstein Hilbert (EH) field equation of gravitation. All these solutions use the well known Christoffel connection:

$$\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu} \tag{16.34}$$

and symmetric metric:

$$g_{\mu\nu} = g_{\nu\mu}. \tag{16.35}$$

So for example:

$$R^{0}{}_{\mu}{}^{\mu 0} \neq 0 \tag{16.36}$$

and summing over repeated μ indices:

$$R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30} \neq 0. \tag{16.37}$$

From anti-symmetry:

$$R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30} \neq 0. \tag{16.38}$$

By definition, the Hodge duals in four dimensions [1–10] of these tensor elements are also anti-symmetric in their last two indices and are defined by:

$$\left.\begin{aligned} \widetilde{R}^{0}{}_{123} &= \parallel g \parallel^{\frac{1}{2}} R^{0}{}_{1}{}^{01} \\ \widetilde{R}^{0}{}_{231} &= \parallel g \parallel^{\frac{1}{2}} R^{0}{}_{2}{}^{02} \\ \widetilde{R}^{0}{}_{312} &= \parallel g \parallel^{\frac{1}{2}} R^{0}{}_{3}{}^{03}. \end{aligned}\right\} \tag{16.39}$$

Therefore Eq. (16.38) is:

$$\widetilde{R}^{0}{}_{123} + \widetilde{R}^{0}{}_{231} + \widetilde{R}^{0}{}_{312} \neq 0 \tag{16.40}$$

in general for exact solutions of the EH field equation of gravitation. Eq. (16.40) is an example of:

$$\widetilde{R}^{\kappa}{}_{\mu\nu\rho} + \widetilde{R}^{\kappa}{}_{\rho\mu\nu} + \widetilde{R}^{\kappa}{}_{\nu\rho\mu} \neq 0 \tag{16.41}$$

in general. Therefore by Eq. (16.22):

$$D_{\mu}\widetilde{T}^{\kappa}_{\nu\rho} + D_{\rho}\widetilde{T}^{\kappa}_{\mu\nu} + D_{\mu}\widetilde{T}^{\kappa}_{\rho\mu} = \widetilde{R}^{\kappa}_{\mu\nu\rho} + \widetilde{R}^{\kappa}_{\rho\mu\nu} + \widetilde{R}^{\kappa}_{\nu\rho\mu} \neq 0 \tag{16.42}$$

which is the same as the equation:

$$D_{\mu}T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0. \tag{16.43}$$

Therefore the right hand side of Eq (16.43) is not zero in general for exact solutions of the EH equation of gravitation, but the left hand side of Eq. (16.43) is always zero for these same exact solutions because they all use the Christoffel symbol (16.34). The torsion tensor $T^{\kappa\mu\nu}$ is always zero for the Christoffel symbol but as we have shown using computer algebra, the curvature tensor is not zero in general for the same Christoffel symbols or connections.

Therefore the EH field equation is self-inconsistent at a fundamental level and must be regarded as obsolete. The reason for the self-inconsistency is that the EH equation was developed in 1915 before the existence of the torsion tensor was realized by Cartan in 1922. The only self-consistent theory of relativity that has been developed and applied to all physics is ECE theory [2–10]. Inferences drawn from the EH theory must be re-evaluated using the correct ECE theory. Example claims of EH theory are Big Bang, the existence of black holes and gravitational radiation, the existence of dark matter, and similar claims that are all based on an incorrect geometry. They cannot therefore be correct physics. Similarly, the so called precision tests of general relativity must be re-evaluated and re-explained with ECE theory, and in this series of papers that re-evaluation has been initiated. Similarly, all known solutions of EH must be tested with the curvature tensor $R^{\kappa}{}_{\mu}{}^{\mu\nu}$, and this is also work in progress. Clearly this is a re-evaluation of a large part of modern physics, the so-called "standard model". The latter can never be a correct unified field theory for these and many other well known reasons [2–10]. Prior to 2007 (paper 93), the curvature tensor $R^{\kappa}{}_{\mu}{}^{\mu\nu}$ was unknown.

In paper 93 it was checked with the same code that for all exact solutions of EH:

$$R^{\kappa}{}_{\mu\nu\sigma} + R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\nu\sigma\mu} = 0 \tag{16.44}$$

This result is known in the now obsolete standard model as "the first Bianchi identity". From Eq. (16.31) it is seen that Eq. (16.44) is equivalent to:

$$R^{a}{}_{b} \wedge q^{b} = 0 \tag{16.45}$$

and therefore incorrectly omits the torsion tensor. Therefore it is geometrically incorrect. It was again developed before the existence of torsion was inferred by Cartan in 1922. It is neither an identity nor was it first inferred by Bianchi. It was actually inferred by Ricci and Levi-Civita. Similarly the so-called "second Bianchi identity" of the standard model:

$$D_{\mu} R^{\kappa}{}_{\sigma\nu\rho} + D_{\rho} R^{\kappa}{}_{\sigma\mu\nu} + D_{\nu} R^{\kappa}{}_{\sigma\rho\mu} = 0 \tag{16.46}$$

is incorrect, again because it incorrectly neglects torsion. Eq. (16.46) in form notation is:

$$D \wedge R^{a}{}_{b} = 0 \tag{16.47}$$

whereas the correct version of Eq. (16.47) must be the derivative identity (16.33). The latter was again inferred (in paper 88) during the development of ECE theory.

An example of Eq. (16.44) is:

$$R^0{}_{123} + R^0{}_{312} + R^0{}_{231} = 0. \tag{16.48}$$

Using Eq. (16.39), Eq. (16.48) is the same as:

$$\widetilde{R}^0{}_1{}^{01} + \widetilde{R}^0{}_2{}^{02} + \widetilde{R}^0{}_3{}^{03} = 0 \tag{16.49}$$

which is an example of:

$$\widetilde{R}^\kappa{}_\mu{}^{\mu\nu} = 0 \tag{16.50}$$

for solutions of the EH field equation of gravitation. Therefore for these solutions:

$$D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu} = 0 \tag{16.51}$$

or in form notation:

$$D \wedge T^a = R^a{}_b \wedge q^b = 0. \tag{16.52}$$

Therefore:

$$\widetilde{T}^{\kappa\mu\nu} = 0, \quad \widetilde{R}^\kappa{}_\mu{}^{\mu\nu} = 0 \tag{16.53}$$

is fortuitously obeyed by solutions of the EH field equation of gravitation, but the Hodge dual, Eq. (16.43) is NOT obeyed by these same solutions of the EH field equation of gravitation. This result means the logical end of the EH theory and was discovered in 2007. Since then ECE theory has been developed to take account correctly of both the Bianchi identity and its Hodge dual identity. These identities provide the basis of the ECE field equations [2–10].

16.3 A New Cyclic Identity of the Cartan Torsion

This identity is inherent in the Bianchi identity in the form (16.22), but has not been hitherto proven. The proof is as follows.

Use the definition of the covariant derivative of a rank three tensor [1–10] to find that:

$$D_\sigma T^\kappa_{\mu\nu} = \partial_\sigma T^\kappa_{\mu\nu} + \Gamma^\kappa_{\sigma\lambda} T^\lambda_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} \Gamma^\kappa_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} \Gamma^\kappa_{\mu\lambda}, \tag{16.54}$$

$$D_\mu T^\kappa_{\nu\sigma} = \partial_\mu T^\kappa_{\nu\sigma} + \Gamma^\kappa_{\mu\lambda} T^\lambda_{\nu\sigma} - \Gamma^\lambda_{\mu\nu} T^\kappa_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} T^\kappa_{\nu\lambda} \tag{16.55}$$

$$D_\nu T^\kappa_{\sigma\mu} = \partial_\nu T^\kappa_{\sigma\mu} + \Gamma^\kappa_{\nu\lambda} T^\lambda_{\sigma\mu} - \Gamma^\lambda_{\nu\sigma} \Gamma^\kappa_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} T^\kappa_{\sigma\lambda} \tag{16.56}$$

Therefore using Eqs. (16.54) to (16.56) in Eq. (16.22):

$$\begin{aligned} D_\sigma T^\kappa_{\mu\nu} &+ D_\mu T^\kappa_{\nu\sigma} + D_\mu T^\kappa_{\sigma\mu} \\ &= (\partial_\sigma T^\kappa_{\mu\nu} + \partial_\mu T^\kappa_{\nu\sigma} + \partial_\nu T^\kappa_{\sigma\mu} + \Gamma^\kappa_{\sigma\lambda} T^\lambda_{\mu\nu} + \Gamma^\kappa_{\mu\lambda} T^\lambda_{\nu\sigma} + \Gamma^\kappa_{\nu\lambda} T^\lambda_{\sigma\mu}) \\ &\quad - (\Gamma^\lambda_{\sigma\mu} T^\kappa_{\lambda\nu} + \Gamma^\lambda_{\mu\sigma} T^\kappa_{\nu\lambda} + \Gamma^\lambda_{\sigma\nu} T^\kappa_{\mu\lambda} + \Gamma^\lambda_{\nu\sigma} T^\kappa_{\lambda\mu} + \Gamma^\lambda_{\mu\nu} T^\kappa_{\lambda\sigma} + \Gamma^\lambda_{\nu\mu} T^\kappa_{\sigma\lambda}) \end{aligned} \tag{16.57}$$

Using the definition of the torsion tensor [1–10]:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \tag{16.58}$$

the first bracket in Eq. (16.57) gives the Bianchi identity (16.3), i.e. gives:

$$\begin{aligned} R^\kappa{}_{\sigma\mu\nu} &+ R^\kappa{}_{\mu\nu\sigma} + R^\kappa{}_{\nu\sigma\mu} \\ &= \partial_\mu \Gamma^\kappa_{\nu\sigma} - \partial_\nu \Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \\ &\quad + \partial_\nu \Gamma^\kappa_{\sigma\mu} - \partial_\sigma \Gamma^\kappa_{\nu\mu} + \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} - \Gamma^\kappa_{\sigma\lambda}\Gamma^\lambda_{\nu\mu} \\ &\quad + \partial_\sigma \Gamma^\kappa_{\mu\nu} - \partial_\mu \Gamma^\kappa_{\sigma\nu} + \Gamma^\kappa_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} \end{aligned} \tag{16.59}$$

Therefore the second bracket in Eq. (16.57) must be identically zero:

$$\Gamma^\lambda_{\sigma\mu} T^\kappa_{\lambda\nu} + \Gamma^\lambda_{\nu\sigma} T^\kappa_{\lambda\mu} + \Gamma^\lambda_{\mu\nu} T^\kappa_{\lambda\sigma} + \Gamma^\lambda_{\sigma\nu} T^\kappa_{\mu\lambda} + \Gamma^\lambda_{\mu\sigma} T^\kappa_{\nu\lambda} + \Gamma^\lambda_{\nu\mu} T^\kappa_{\sigma\lambda} = 0. \tag{16.60}$$

Now use the anti-symmetry:

$$T^\kappa_{\lambda\mu} = -T^\kappa_{\mu\lambda} \tag{16.61}$$

to prove a new cyclic identity obeyed by the Cartan torsion:

$$T^\kappa_{\lambda\nu} T^\lambda_{\sigma\mu} + T^\kappa_{\lambda\mu} T^\lambda_{\nu\sigma} + T^\kappa_{\lambda\sigma} T^\lambda_{\mu\nu} = 0 \tag{16.62}$$

In form notation Eq. (16.62) is the wedge product:

$$T^\kappa{}_\lambda \wedge T^\lambda = 0 \tag{16.63}$$

which in short-hand notation may be denoted:

$$T \wedge T = 0. \tag{16.64}$$

Here $T^\kappa{}_\lambda$ is defined as the tensor valued one-form [1–10] of index ν:

$$T^\kappa{}_\lambda := T^\kappa{}_{\lambda\nu} \tag{16.65}$$

and T^λ is defined as the vector valued two-form of index $\sigma\mu$:

$$T^\lambda := T^\lambda{}_{\sigma\mu} \,. \tag{16.66}$$

The short-hand notation (16.64) explains the basic structure of the tensor identity (16.62) as being akin, approximately writing, to the cross product of a vector with itself, or akin to the Poincare Lemma in structure. The new theorem (16.62) is inherent in the Bianchi identity itself and is another way of revealing the rigorous self-consistency of Cartan geometry. If an attempt is made to assert a geometry that is not consistent with the Cartan geometry, the inconsistency will sooner or later reveal itself. In the case of EH theory it finally revealed itself through the curvature tensor $R^\kappa{}_\mu{}^{\mu\nu}$. The latter is essentially impossible to compute by hand, because of the intricate complexity of its internal structure (see paper 93 of www.aias.us). Therefore $R^\kappa{}_\mu{}^{\mu\nu}$ was never computed prior to 2007, and was obviously unknown to Einstein himself. It was also unknown to the instigators of Big Bang theory, notably Hawking and Penrose, and the instigator of black hole theory, Wheeler. Therefore the standard model claims have become elaborate, but are all based on a fundamentally incorrect geometry. The ECE theory is therefore the only self-consistent theory of relativity.

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