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Derivation of the Thomas Precession in Terms of the Infinitesimal Torsion Generator

by

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Abstract

The Thomas precession is shown to be due to the rotation of Minkowski space-time, a rotation which is described by the infinitesimal generator of the Cartan torsion. Rotation of the Minkowski space-time defines the Thomas angular velocity and the Thomas time dilation observed in spin-orbit coupling in atomic and molecular spectroscopy. It is shown that the Thomas angular velocity is generated directly by the commutator of two Lorentz boost generators. The infinitesimal Cartan torsion generator for a rotation about the Z axis is shown to be the time derivative of the infinitesimal rotation generator of the Lorentz or Poincaré groups. Within a factor $\hbar$ this is the angular momentum operator of quantum mechanics. Therefore the Cartan torsion generator is observed throughout quantum physics. Thomas precession is observed in pendulum precession and is due to the infinitesimal generator of Cartan torsion.

Keywords: Thomas precession, infinitesimal generator of Cartan torsion, ECE theory.

17.1 Introduction

In the well known differential geometry of Cartan [1–11] the torsion is derived by the first structure equation in terms of the tetrad and spin connection. The curvature is defined by the second structure equation in terms of the spin connection. Finally the torsion and curvature are linked by the development of the Bianchi identity by Cartan. This is a self-consistent geometry based on the existence of space-time torsion and curvature. It is incorrect to assert arbitrarily that torsion vanishes, so the meaning of Cartan torsion in relativity must be evaluated systematically. This has been done in the Einstein Cartan Evans series of papers on a suggested unified field theory based on the principle of general covariance (see ECE papers on the www.aias.us website). In this paper the well known Thomas precession is shown to be due to the infinitesimal generator of a Cartan torsion caused by the rotation of the Minkowski frame. In Section 17.2 the Thomas angular velocity and proper time are obtained straightforwardly by rotating the Minkowski metric. In a static Minkowski space-time all Christoffel connections are zero by definition, so there is no Christoffel torsion, no Cartan torsion, no Riemann curvature and no Cartan curvature. On rotation of the Minkowki frame however, a Cartan torsion and non-zero connection are generated as shown in Section 17.3. This Cratan torsion due to rotation is the time derivative of the rotation matrix, and so it is possible to define the infinitesimal generator of Cartan torsion. By considering the commutator of two Lorentz boost generators, it is shown that this generator of torsion is produced directly by the commutator, together with the Thomas angular velocity. The Thomas factor is produced by time dilation (again introduced by rotation of Minkowski space-time) and is observed in spin orbit coupling in atomic and molecular spectroscopy. The latter observes the generator of torsion directly as the angular momentum operator of quantum mechanics, which within h is the infinitesimal rotation generator. The infinitesimal generator of Cartan torsion is therefore proportional directly to the angular momentum operator [12] of quantum mechanics.

17.2 Rotation of the Minkowski Space-Time

The static Minkowski line element in cylindrical polar co-ordinates is well known to be:

$$ds^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dZ^2. \tag{17.1}$$

The cylindrical polar and Cartesian coordinates are related by:

$$\left.\begin{aligned} X &= r\cos\phi, \\ Y &= r\sin\phi, \\ Z &= Z. \end{aligned}\right\} \tag{17.2}$$

Here c is the speed of light and t the time. Now consider a Minkowski frame denoted by:

$$ds'^2 = c^2dt^2 - dr^2 - r^2d\phi'^2 - dZ^2 \tag{17.3}$$

where:

$$\phi' = \phi + \omega t \tag{17.4}$$

represents a rotation of the ϕ coordinate defined by the angular velocity ω. Therefore the infinitesimals are related by:

$$d\phi' = d\phi + \omega dt \tag{17.5}$$

and the squares of the infinitesimals are related by:

$$d\phi'^2 = d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2. \tag{17.6}$$

Therefore the rotating Minkowski line element is:

$$ds'^2 = (c^2 - r^2\omega^2)dt^2 - 2\omega r^2 d\phi dt - dr^2 - r^2 d\phi^2 - dZ^2 \tag{17.7}$$

Now define the linear velocity v by:

$$v = r\omega \tag{17.8}$$

to find that the rotating line element is:

$$ds'^2 = \left(1 - \frac{v^2}{c^2}\right)(c^2 dt^2 - 2r^2 \Omega d\phi dt) - dr^2 - r^2 d\phi^2 - dZ^2 \tag{17.9}$$

where:

$$\Omega = \omega \left(1 - \frac{v^2}{c^2}\right)^{-1} \tag{17.10}$$

is a relativistically corrected angular velocity. It is known as the Thomas angular velocity and is produced directly by the rotation of the Minkowski space-time. The infinitesimal of time is changed by this rotation to:

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \tag{17.11}$$

and this is the time dilation effect responsible for the well known Thomas factor in spin-orbit coupling. The effects of spin orbit coupling are well known in atomic and molecular spectroscopy. Finally the Thomas precession observed in a pendulum for a rotation of 2π radians:

$$\omega dt = 2\pi \tag{17.12}$$

is defined as:

$$\alpha = \Omega d\tau - \omega dt = 2\pi \left(\left(1 - \frac{v^2}{c^2} \right)^{1/2} - 1 \right). \tag{17.13}$$

These are well known effects demonstrated straightforwardly by the simple rotation of a Minkowski space-time as defined in Eq. (17.4). The original demonstrations by Thomas in 1927 and Sommerfeld in 1931 are much more complicated.

17.3 Infinitesimal Generator of the Cartan Torsion

In this section it is shown that the well known phenomena derived in Section 17.2 are due to the Cartan torsion, specifically the infinitesimal torsion generator. Consider the rotation of a three-vector about the Z axis [13, 14]:

$$\begin{bmatrix} V'_X \\ V'_Y \\ V'_Z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} \tag{17.14}$$

where θ is the angle of rotation. The tetrad is always defined in Cartan geometry as the matrix linking two vectors, so the tetrad for this rotation is defined as:

$$q^a_\mu = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17.15}$$

The angular velocity is defined as:

$$\omega = \frac{d\theta}{dt}. \tag{17.16}$$

The spin connection for rotation is the matrix representation of the tetrad [1–11]:

$$\omega^a{}_b = \frac{\omega}{c} \epsilon^a{}_{bc} q^c \tag{17.17}$$

where $\epsilon^a{}_{bc}$ is the totally anti-symmetric unit tensor in three dimensions. So the Cartan torsion for this rotation:

$$T^a_{\mu\nu} = \partial_\mu q^a_\mu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \tag{17.18}$$

is fully defined. The elements of the tetrad are:

$$\left.\begin{aligned} q_1^1 = \cos\theta, q_2^1 = \sin\theta, q_3^1 = 0, \\ q_1^2 = -\sin\theta, q_2^2 = \cos\theta, q_3^2 = 0, \\ q_1^3 = 0, q_2^3 = 0, q_3^3 = 1. \end{aligned}\right\} \tag{17.19}$$

These are components of tetrad vectors in the Cartesian coordinate system:

$$\left.\begin{aligned} \boldsymbol{q}^1 = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \\ \boldsymbol{q}^2 = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}, \\ \boldsymbol{q}^3 = \boldsymbol{k}. \end{aligned}\right\} \tag{17.20}$$

The spin connection and tetrad components are related by:

$$\left.\begin{aligned} \omega^1{}_2 = \frac{\omega}{c} q^3, \\ \omega^2{}_3 = \frac{\omega}{c} q^1, \\ \omega^3{}_1 = \frac{\omega}{c} q^2. \end{aligned}\right\} \tag{17.21}$$

Therefore the non-zero spin connection components are:

$$\left.\begin{aligned} \omega^2{}_{13} = \frac{\omega}{c} q_1^1, \omega^3{}_{11} = \frac{\omega}{c} q_1^2, \\ \omega^2{}_{23} = \frac{\omega}{c} q_2^1, \omega^3{}_{21} = \frac{\omega}{c} q_2^2, \\ \omega^1{}_{32} = \frac{\omega}{c} q_3^3. \end{aligned}\right\} \tag{17.22}$$

The angle θ depends on time as in Eq. (17.16), which defines the angular velocity. Therefore the non-zero torsion elements are:

$$T^a_{0\mu} = \frac{1}{c}\frac{\partial q^a_\mu}{\partial t} = \frac{\omega}{c}\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{17.23}$$

Define the infinitesimal generator of the Cartan torsion by:

$$T_Z = \frac{1}{i}\frac{\omega}{c}\begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\theta\to 0} \tag{17.24}$$

to find that it is directly proportional to the well known infinitesimal rotation generator:

$$T_Z = \frac{\omega}{c} J_Z \tag{17.25}$$

where:

$$J_Z = -i \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17.26}$$

The angular momentum operator of quantum mechanics is the rotation generator within a factor $\hbar$, the reduced Planck constant. Therefore the infinitesimal generator of torsion is directly proportional to the angular momentum operator which is basically important to all quantum mechanics [1–14]. Therefore the Cartan torsion is observed in all quantum mechanics.

It is also well known 1-14 that the Lorentz and Poincaré groups contain commutator equations such as:

$$[\kappa_X, \kappa_Y] = -iJ_Z \tag{17.27}$$

where κ_X and κ_Y are Lorentz boost generators. Lorentz boosts in different directions cause a rotation, and this is the origin of the Thomas factor and Thomas precession. In Section 2 the Minkowski frame was rotated to illustrate these relativistic phenomena in a simple way. Consider a Lorentz boost in the X axis:

$$\begin{aligned} &X' = \gamma(X + vt),\ Y' = Y, t' = \gamma(t + vX/c^2), \\ &\gamma = (1 - v^2/c^2)^{-1/2},\ \beta = v/c,\ X^0 = ct,\ X' = X, \\ &r = \cosh\phi, \gamma\beta = \sinh\phi, \frac{v}{c} = \tanh\phi. \end{aligned} \tag{17.28}$$

The Lorentz boost in matrix form is therefore:

$$\begin{bmatrix} X^{0'} \\ X^{1'} \\ X^{2'} \\ X^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{bmatrix} \tag{17.29}$$

where the matrix in Eq. (17.29) is a Lorentz boost matrix. This is also a type of tetrad in Cartan geometry. Therefore in Cartan geometry the group generators of the well known Lorentz and Poincaré groups of special relativity are all tetrads. Similarly the Lorentz boost matrix in Y is [1–14]:

$$B_Y = \begin{bmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & \gamma & 0 & 0 \\ \gamma\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17.30}$$

The infinitesimal boost generators are:

$$\begin{aligned} \kappa_X &= -i\frac{\partial B_X}{\partial \phi}|_{\phi\to 0} = -i\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \kappa_Y &= -i\frac{\partial B_Y}{\partial \phi}|_{\phi\to 0} = -i\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{17.31}$$

and the infinitesimal rotation generator is:

$$J_Z = -i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17.32}$$

These generators give Eq. (17.27).

It is found that:

$$[B_X, B_Y] = i\left(\frac{v}{c}\right)^2\left(1-\frac{v^2}{c^2}\right)^{-1} J_Z. \tag{17.33}$$

Therefore the relativistically corrected infinitesimal rotation generator is:

$$J'_Z = \left(\frac{v}{c}\right)^2\left(1-\frac{v^2}{c^2}\right)^{-1} J_Z. \tag{17.34}$$

Therefore the relativistically corrected, infinitesimal torsion generator is:

$$T'_Z = \left(\frac{v}{c}\right)^2\frac{\Omega}{c}J_Z \tag{17.35}$$

where:

$$\Omega = \omega \left(1 - \frac{v^2}{c^2}\right)^{-1} \tag{17.36}$$

is the Thomas angular velocity (17.10). The latter has therefore been derived directly from the commutator of Lorentz boosts as in Eq. (17.33), and it has been proven that the Thomas angular velocity is part of the relativistic correction to the infinitesimal torsion generator.

Finally the tetrad postulate [1–11]:

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Gamma^\lambda_{\mu\sigma} q^a_\lambda = 0 \tag{17.37}$$

is used to show that there exists a precessional equation:

$$\partial_0 q^1_1 = \Gamma^0{}_{01} q^1_1 + \Gamma^2{}_{01} q^1_2 \tag{17.38}$$

directly from Cartan geometry. A particular solution of Eq. (17.38) is:

$$\Gamma^2{}_{01} = -\frac{\omega}{c},\ \Gamma^1{}_{01} = 0. \tag{17.39}$$

Note carefully that this is not a Christoffel connection, because in this example:

$$\Gamma^2{}_{10} = 0 \tag{17.40}$$

and the Christoffel connection must be symmetric in its lower two indices:

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}. \tag{17.41}$$

The Christoffel torsion:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \tag{17.42}$$

is always zero, and the Cartan torsion defined by the Christoffel torsion:

$$T^a_{\mu\nu} = q^a{}_\kappa T^\kappa_{\mu\nu} \tag{17.43}$$

is always zero. Therefore the gamma connection of type (17.39) is a rotational gamma connection defined by a non-zero Cartan torsion introduced by rotating the Minkowski frame. It is not a Christoffel connection.

This relatively straightforward analysis of a rotating Minkowski frame can be applied to rotate classes of line elements that are exact solutions of the

Einstein Hilbert (EH) field equation. It was discovered in 2007 (paper 93 of www.aias.us) that the geometry of the EH field equation is incorrect due to its neglect of torsion. In subsequent papers the torsion was used self-consistently, and this paper (paper 110 of the ECE series) it has been used to describe the simplest rotational phenomenon of special relativity - the Thomas precession. Other well known precessional effects include the Lense Thirring rotational frame dragging, and in subsequent papers it will be shown that the Lense Thirring effect is also based on the Cartan torsion, it is effectively a rotation of a line element that is a solution of the EH field equation. The conventional or standard model explanation of the Lense Thirring effect is now known to be incorrect, because it still uses a Christoffel connection in which torsion is zero.

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