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# *THE ELECTROSTATIC AND MAGNETOSTATIC FIELDS GENERATED BY LIGHT IN FREE SPACE**

## I. INTRODUCTION

The phenomenological equations of J. C. Maxwell form the basis of the classical understanding of light. The equations were formulated in the mid nineteenth century, before relativity was fully developed, and before the quantum theory came into existence. They were later put on a microscopic basis by H. A. Lorentz in his theory of the electron, and have become the starting point of a vast number of contemporary papers on the nature of light in free space and in materials. In this paper we show that there exist novel electro and magnetostatic fields in the propagation axis of the classical electromagnetic plane wave, fields that propagate in free space and conserve the structure of the well-defined Poynting vector, and therefore do not affect the law of conservation of electromagnetic energy in free space. It is usually assumed that the following are solutions to the free space Maxwell equations for a completely circularly polarized plane wave:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{\sqrt{2}} E_0(\mathbf{i} + i\mathbf{j})e^{i\phi} \tag{1}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{\sqrt{2}} B_0(\mathbf{j} - i\mathbf{i})e^{i\phi} \tag{2}$$

for suggesting the possibility of $\mathbf{E}_\Pi$.

Here $E_0$ is the scalar electric field strength amplitude, and $B_0$ the scalar magnetic flux density amplitude, **i** and **j** are unit vectors in $X$ and $Y$ of the laboratory frame, and $\phi$ is the phase of the plane wave. These solutions are oscillatory and time and space dependent through the phase

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r} \tag{3}$$

where $\omega$ is the angular frequency of the wave, $t$ the time, $\boldsymbol{\kappa}$ the wave vector, and **r** a position vector as usual. A whole literature is available concerning their properties.

However, the equations

$$\mathbf{E}^{G} = \mathbf{E}(\mathbf{r}, t) + \mathbf{E}_\Pi \tag{4}$$

$$\mathbf{B}^{G} = \mathbf{B}(\mathbf{r}, t) + \mathbf{B}_\Pi \tag{5}$$

are also valid solutions to the free space Maxwell equations. Here $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are uniform, time-independent, electric and magnetic fields directed in the propagation axis $Z$ of the plane wave. It appears always to have been implicitly assumed that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are both zero in free space, and that there is no component in $Z$ of the plane wave in vacuo. There is no mathematical reason for this supposition, however, and as we shall see, the vectors $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ can be related to the well-known $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. The source of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ is therefore the same as the source of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. If the latter are nonzero, then so are both $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$, in general.

Section II introduces $\mathbf{B}_\Pi$ using the imaginary conjugate product,

$$\boldsymbol{\Pi}^{(A)} \equiv E_0 c \, \mathrm{Im}(\mathbf{B}_\Pi) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{E}^*(\mathbf{r}, t) = -iE_0^2\mathbf{k} \tag{6}$$

of the electromagnetic plane wave,[1–8] where $\mathbf{E}^*(\mathbf{r}, t)$ is the complex conjugate of $\mathbf{E}(\mathbf{r}, t)$, i.e.,

$$\mathbf{E}^*(\mathbf{r}, t) = \frac{1}{\sqrt{2}} E_0(\mathbf{i} - i\mathbf{j})e^{-i\phi} \tag{7}$$

We see in Appendix A that the real and imaginary parts of $\mathbf{B}_\Pi$ are the same.

The law of conservation of energy for a plane wave in free space can be expressed through the continuity equation:

$$\nabla \cdot \mathbf{N} = -\frac{\partial U}{\partial t} \tag{8}$$

where $\mathbf{N}$ is the Poynting vector:

$$\mathbf{N} = \frac{1}{\mu_0}\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \tag{9}$$

and $U$ a scalar field. Here $\mu_0$ is the magnetic permeability of free space. The vector $\mathbf{N}$ is the flux of electromagnetic energy of the plane wave, and the scalar $U$ is the electromagnetic field's energy density. Therefore, $\mathbf{N}$ is electromagnetic power per unit area, and $U$ is power per unit volume. The scalar amplitude of the Poynting vector is the light intensity $I_0$. Therefore, Eq. (8) expresses, in classical electrodynamics, the law of conservation of electromagnetic energy in free space. This idea of field energy has no meaning[9] unless the wave interacts with matter (e.g., an electron). In Section IV, it is shown that the continuity equation (8) is unchanged for nonzero $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ provided they are both complex and

$$\mathbf{B}_{\Pi} \times \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_{\Pi} \times \mathbf{B}(\mathbf{r}, t) \tag{10}$$

In other words, Eq. (10) is the condition for conservation of free space electromagnetic energy given the general solutions (4) and (5) of the Maxwell equations. Equation (10) shows that if $\mathbf{B}_{\Pi}$ is real, and defined through the conjugate product (6), then $\mathbf{E}_{\Pi}$ is imaginary. Finally, a discussion is given of the physical meaning of the novel vectors $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$, with order of magnitude estimates, and experimental consequences.

## II. THE DEFINITION OF $\mathbf{B}_{\Pi}$ THROUGH THE CONJUGATE PRODUCT

The conjugate product $\mathbf{E} \times \mathbf{E}^*$ appears in the antisymmetric part of Maxwell's stress tensor[10] and is a well-defined property of light. It is an axial vector with magnetic symmetry,[11, 12] i.e., that of angular momentum: positive to parity inversion $\hat{P}$, and negative to motion reversal $\hat{T}$. The vector notation $\mathbf{E} \times \mathbf{E}^*$ is equivalent to the tensor notation

$$\Pi_i^{(A)} = \tfrac{1}{2}\varepsilon_{ijk}(E_j E_k^* - E_k E_j^*) \tag{11}$$

where $\varepsilon_{ijk}$ is the Levi-Civita symbol. This shows that the axial vector $\mathbf{E} \times \mathbf{E}^*$ is equivalent to a polar rank two tensor:

$$\Pi_{jk}^{(A)} = \tfrac{1}{2}(E_j E_k^* - E_k E_j^*) \tag{12}$$

which is the antisymmetric part of the tensor $E_j E_k^*$. Therefore, $\mathbf{E} \times \mathbf{E}^*$ is the vector part of light intensity.

The quantity

$$\mathrm{Im}(\mathbf{B}_{\Pi}) = \frac{\mathrm{i}\mathbf{E} \times \mathbf{E}^*}{E_0 c} \tag{13}$$

is a uniform, divergentless, time-independent, magnetic flux density vector with the required symmetry and units. The magnetic field $\mathbf{B}_{\Pi}$ exists in free space because $\mathbf{E} \times \mathbf{E}^*$ exists in free space, and is defined in the $Z$ axis:

$$\mathrm{Im}(\mathbf{B}_{\Pi}) = +\frac{E_0}{c}\mathbf{k} = +B_0\mathbf{k} \tag{14}$$

where $\mathbf{k}$ in axial unit vector. The magnitude of $\mathbf{B}_{\Pi}$, i.e., $|\mathbf{B}_{\Pi}|$, is the scalar amplitude $B_0$ defined in the introduction. A real interaction Hamiltonian is produced from $\mathbf{E} \times \mathbf{E}^*/(E_0 c)$ when it forms a scalar product with the usual imaginary magnetic dipole moment operator, $\mathrm{i}\hat{\mathbf{m}}''$, in quantum mechanics.[13–15] Similarly, the imaginary $\mathbf{E} \times \mathbf{E}^*$ produces a well-defined[1–8] real interaction Hamiltonian when it multiplies the imaginary part of molecular electric polarizability operator, $\mathrm{i}\hat{\boldsymbol{\alpha}}''$. The latter is the vectorial polarizability,[16, 17] which vanishes at zero frequency from time-dependent perturbation theory. Both $\hat{\mathbf{m}}''$ and $\hat{\boldsymbol{\alpha}}''$ are directly proportional (using the Wigner-Eckart theorem, for example,[16, 17] to the net molecular electronic angular momentum operator $\hat{J}$:

$$\hat{\mathbf{m}}'' = \gamma_e \hat{J} \tag{15}$$

$$\hat{\boldsymbol{\alpha}}'' = \gamma_{\Pi} \hat{J} \tag{16}$$

where $\gamma_e$ is the gyromagnetic ratio[13–15] and $\gamma_{\Pi}$ is the gyroptic ratio.[18–20] Consequently,

$$\hat{\boldsymbol{\alpha}}'' = \frac{\gamma_{\Pi}}{\gamma_e}\hat{\mathbf{m}}'' \tag{17}$$

showing that $\hat{m}''$ and $\hat{\alpha}''$ have the same $\hat{T}$-negative, $\hat{P}$-positive symmetry, and are both axial vector operators. The conjugate product $\mathbf{E} \times \mathbf{E}^*$ forms a real Hamiltonian operator when it multiplies $\mathrm{i}\hat{\boldsymbol{\alpha}}''$, and because $\hat{\boldsymbol{\alpha}}''$ is directly proportional to $\hat{J}$ and thus to $\hat{\mathbf{m}}''$, it follows that $\mathbf{E} \times \mathbf{E}^*$ must be proportional to a magnetic field, which we have identified as $\mathbf{B}_{\Pi}$ in Eq. (13). Clearly, $\hat{\mathbf{m}}''$ can form a real Hamiltonian operator only when multiplied by a magnetic field. The root of Eq. (13) is therefore found in the fact that the molecular property tensors $\hat{\boldsymbol{\alpha}}''$ *and* $\hat{\mathbf{m}}''$ are both axial vectors with magnetic symmetry. This point can be emphasized by assuming that the real part of $\hat{\mathbf{m}}'' \cdot \mathbf{B}_\Pi$ is an interaction Hamiltonian and investigating the logical consequences. To do this, it is convenient to write the interaction Hamiltonian[16, 17] between $\mathrm{i}\hat{\boldsymbol{\alpha}}''$ and $\mathbf{E} \times \mathbf{E}^*$ as

$$\Delta \hat{H} = -\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^* = -\mathrm{i}E_0 c \hat{\boldsymbol{\alpha}}'' \cdot \frac{\mathbf{E} \times \mathbf{E}^*}{E_0 c} \equiv -E_0 c \hat{\boldsymbol{\alpha}}'' \mathrm{Im}(\mathbf{B}_\Pi) \quad (18)$$

where we have defined $\mathbf{B}_\Pi$ in terms of $\mathbf{E} \times \mathbf{E}^*$ as in Eq. (13). Using the proportionality (17) between the magnetic dipole moment and the vectorial polarizability, Eq. (18) becomes

$$\Delta \hat{H} = -E_0 c \frac{\gamma_\Pi}{\gamma_e} \hat{\mathbf{m}}'' \cdot \mathrm{Im}(\mathbf{B}_\Pi) = -\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^* \quad (19)$$

showing that the product $\hat{\mathbf{m}} \cdot \mathbf{B}_\Pi$ is directly proportional to the product $\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^*$ through a nonzero proportionality constant. Therefore, if the energy $\mathrm{i}\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^*$ is nonzero, then so must the energy $\hat{\mathbf{m}}'' \cdot \mathbf{B}_\Pi$ be nonzero.

Using the Wigner-Eckart theorem, the gyromagnetic and gyroptic ratios can be defined as follows in an atom with net electronic angular momentum, $\hat{J}$, showing that in this case $\gamma_e$ and $\gamma_\Pi$ are nonzero in general:

$$m_0^{1''} = \frac{\langle J \| \hat{m}^{1''} \| J \rangle}{\langle J \| \hat{J}^1 \| J \rangle} J_0^1 = \gamma_e J_0^1 \quad (20)$$

$$\alpha_0^{1''} = \frac{\langle J \| \hat{\alpha}^{1''} \| J \rangle}{\langle J \| \hat{J}^1 \| J \rangle} J_0^1 = \gamma_\Pi J_0^1 \quad (21)$$

$$\langle J \| \hat{J}^1 \| J \rangle = \left[ J(J+1)(2J+1) \right]^{1/2} \quad (22)$$

## III. THE CONSERVATION OF ELECTROMAGNETIC ENERGY

We have assumed that Eqs. (4) and (5) are solutions of the free space Maxwell equations:

$$\nabla \times \mathbf{E}^G = -\frac{\partial \mathbf{E}^G}{\partial t} \quad (23)$$

$$\nabla \times \mathbf{B}^G = \frac{1}{c^2} \frac{\partial \mathbf{E}^G}{\partial t} \quad (24)$$



in SI units. It is clear that if $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are defined in the $Z$ axis of the plane wave, then

$$\nabla \times \mathbf{E}_\Pi = \frac{\partial \mathbf{B}_\Pi}{\partial t} = 0 \quad (25)$$

$$\nabla \times \mathbf{B}_\Pi = \frac{\partial \mathbf{E}_\Pi}{\partial t} = 0 \quad (26)$$

because these fields are time independent and have no $X$ and $Y$ components. Consider the divergence of $\mathbf{E}^G(\mathbf{r}, t)$ and $\mathbf{B}^G(r, t)$. Using the vector identity,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (27)$$

it follows that we may expand:

$$\begin{aligned} \nabla \cdot (\mathbf{E}^G \times \mathbf{B}^G) = {} & \nabla \cdot (\mathbf{E} \times \mathbf{B}) + \nabla \cdot (\mathbf{E} \times \mathbf{B}_\Pi) \\ & + \nabla \cdot (\mathbf{E}_\Pi \times \mathbf{B}) + \nabla \cdot (\mathbf{E}_\Pi \times \mathbf{B}_\Pi) \end{aligned} \quad (28)$$

where $\mathbf{E} \times \mathbf{B}$ is proportional to the Poynting vector of the law of conservation of energy, Eq. (8). From the relations

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{B}_\Pi) &= \mathbf{B}_\Pi \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}_\Pi) \\ \nabla \times \mathbf{B}_\Pi &= 0 \end{aligned} \quad (29)$$

and

$$\mathbf{B}_\Pi \cdot (\nabla \times \mathbf{E}) = -\mathbf{B}_\Pi \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (30)$$

it follows that

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}_\Pi) = 0 \quad (31)$$

and similarly

$$\nabla \cdot (\mathbf{E}_\Pi \times \mathbf{B}) = 0 \quad (32)$$

Also, the last term in Eq. (28) vanishes because $\mathbf{E}_\Pi$ is parallel to $\mathbf{B}_\Pi$ in $Z$. It follows, therefore, that

$$\nabla \cdot (\mathbf{E}^G \times \mathbf{B}^G) = \nabla \cdot (\mathbf{E} \times \mathbf{B}) \quad (33)$$

i.e., the continuity equation (8) is unaffected by the presence of $\mathbf{E}_\Pi$ and

$\mathbf{B}_{\Pi}$, and the relation (Eq. (8)) of the field energy flux density ($\mathbf{N}$) to the electromagnetic field energy density ($U$) is unchanged in the free space electromagnetic plane wave. In other words, the electromagnetic powers per unit area generated by $\nabla \cdot (\mathbf{E} \times \mathbf{B}_{\Pi})$ and by $\nabla \cdot (\mathbf{E}_{\Pi} \times \mathbf{B})$ are both zero, and therefore so are the associated electromagnetic powers per unit volume.

This result is true only if $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ are both in the propagation axis of the plane wave. The argument so far shows that $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ may be separately non zero, or that $\mathbf{E}_{\Pi}$ may be zero and $\mathbf{B}_{\Pi}$ nonzero, as defined in Eq. (13).

To obtain a relation between $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ we use the result, from Eq. (8):

$$\nabla \cdot (\mathbf{E}^{G} \times \mathbf{B}^{G}) = \nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{\partial U}{\partial t} \tag{34}$$

which implies that the divergence of the product $\mathbf{E}^{G} \times \mathbf{B}^{G}$ is nonzero and identical with the divergence of the product $\mathbf{E} \times \mathbf{B}$. This implies that

$$\mathbf{E}^{G} \times \mathbf{B}^{G} = \mathbf{E} \times \mathbf{B} + \text{constant} \tag{35}$$

However, we know that

$$\mathbf{E}^{G} \times \mathbf{B}^{G} = \mathbf{E} \times \mathbf{B} + \mathbf{E}_{\Pi} \times \mathbf{B} + \mathbf{E} \times \mathbf{B}_{\Pi} \tag{36}$$

and from Eqs. (35) and (36) we derive the key result:

$$\mathbf{E}_{\Pi} \times \mathbf{B} = \mathbf{B}_{\Pi} \times \mathbf{E} \tag{37}$$

assuming that the constant of integration in Eq. (35) is zero (see Appendix D) and demonstrating that if $\mathbf{B}_{\Pi}$ is real, then $\mathbf{E}_{\Pi}$ must be imaginary.

In precise analogy with $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, the fields $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ take meaning only when there is wave–particle or wave–matter interaction, but these fields propagate through free space (i.e., vacuum). Clearly, electromagnetic waves can be detected only when there is particulate matter with which the waves can interact, otherwise there would be no experimental evidence at all for the existence of electromagnetic fields. The source of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ is the same as the source of the oscillating fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, because both the static and oscillating components are needed for the complete solution of the free space Maxwell equations and the presence of oscillating components implies through Eqs. (13) and (37) the presence of static components. The static and oscillating components are



both relativistic in nature, because the plane wave propagates at the speed of light. In quantum field theory, there are operator equivalents[18–21] of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$. A fundamentally important difference between the oscillating and static components of the plane wave is that the former vanish upon time averaging and the latter do not. This is the source of several novel physical phenomena when there is wave–matter interaction. Equation (37) conserves $\hat{P}$ and $\hat{T}$ symmetry, and the static components of the solution are related through Eqs. (13) and (37) to the oscillating components with, as we have seen, conservation of electromagnetic energy. The components are therefore completely defined and the definition is self-consistent.

From the properties of the dual transform of special relativity (see Appendix A) the Maxwell equations are invariant to

$$\begin{aligned} c\mathbf{B} &\rightarrow -\mathrm{i}\mathbf{E} \\ c\mathbf{B}_{\Pi} &\rightarrow -\mathrm{i}\mathbf{E}_{\Pi} \end{aligned} \tag{38}$$

The dual transform implies immediately that

$$\mathbf{B}_{\Pi} \times \mathbf{E} = -\mathrm{i}\frac{\mathbf{E}_{\Pi}}{c} \times \left(-c\frac{\mathbf{B}}{\mathrm{i}}\right) = \mathbf{E}_{\Pi} \times \mathbf{B} \tag{39}$$

which confirms that the sum

$$\mathbf{E}_{\Pi} \times \mathbf{B} + \mathbf{E} \times \mathbf{B}_{\Pi} = 0 \tag{40}$$

and that the general solution of Maxwell's equations must be of the form (see Appendix A)

$$\mathbf{E}^{G} = \mathbf{E}(\mathbf{r}, t) \pm E_0(\mathrm{i} - 1)\mathbf{k} \tag{41}$$

$$\mathbf{B}^{G} = \mathbf{B}(\mathbf{r}, t) \pm B_0(\mathrm{i} + 1)\mathbf{k} \tag{42}$$

to be consistent with the theory of special relativity applied to the Maxwell equations.

It is easily checked that Eq. (39) is consistent with Eqs. (1) and (2), with

$$\mathbf{B}_{\Pi} = \pm B_0(\mathrm{i} + 1)\mathbf{k} \tag{43}$$

$$\mathbf{E}_{\Pi} = \pm E_0(\mathrm{i} - 1)\mathbf{k} \tag{44}$$

Equation (39) is also consistent with the generalized continuity equation, and with the fact (see appendix A) that

$$\mathbf{F}_{\Pi} = \mathbf{E}_{\Pi} + \mathrm{i}c\mathbf{B}_{\Pi} \tag{45}$$

and

$$F_{\Pi}^2 = E_{\Pi}^2 - c^2 \mathbf{B}_{\Pi}^2 + 2ic\mathbf{E}_{\Pi} \cdot \mathbf{B}_{\Pi} \tag{46}$$

are invariants of the Lorentz transform. From Eq. (40), the net contribution of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ to free space electromagnetic energy is zero.

## IV. DISCUSSION

The orders of magnitude of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ can be estimated directly from the intensity $I_0$ of the light beam in W m$^{-2}$, through the free space relations

$$|\mathbf{B}_{\Pi}| = B_0 = \left(\frac{I_0}{\varepsilon_0 c^3}\right)^{1/2} \qquad |\mathbf{E}_{\Pi}| = E_0 = \left(\frac{I_0}{\varepsilon_0 c}\right)^{1/2} \tag{47}$$

where $\varepsilon_0$ is the electric permittivity in vacuo ($8.854 \times 10^{-12}$ J$^{-1}$ C$^2$m$^{-1}$ in S.I. units). Thus, for a beam of 10 000 W m$^{-2}$, (1.0 W cm$^{-2}$), $B_0$ is about $10^{-8}$ T and $E_0$ about 20 V m$^{-1}$. These are also the scalar amplitudes $B_0$ and $E_0$ of the oscillating part of the solution to Maxwell's equations, and the scalar intensity $I_0$ of the beam is unaffected by the presence of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ because $I_0$ is the magnitude of the Poynting vector. However, $\mathbf{B}_{\Pi}$ is nonzero after time averaging because it is independent of time, and forms a real, nonzero, interaction Hamiltonian with particulate matter. This Hamiltonian leads, therefore, to the prediction of novel physical phenomena, which can be measured as a function of $I_0$ and of the polarization state of the light beam. If the latter is linearly or incoherently polarized, $\mathbf{E} \times \mathbf{E}^*$ is zero and in consequence, so are $\mathbf{B}_{\Pi}$ and $\mathbf{E}_{\Pi}$; otherwise $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ are proportional to the square root of $I_0$. Because $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ are electrostatic and magnetostatic fields that form part of the general solution of Maxwell's equations in free space, they have the properties of such fields when light interacts with matter. This is the main conclusion of this paper.

On the basis of this conclusion it is easy to see that the various theories of the interaction of conventionally generated electric and magnetic fields can be applied directly to the real field $\mathbf{B}_{\Pi}$, and examples of these applications have been given elsewhere for $\mathbf{B}_{\Pi}$ (Refs. 22–26). These include the inverse Faraday effect, the optical Faraday and Zeeman effects, optically induced shifts in NMR resonances ("optical NMR", recently observed experimentally,[27] the optical Cotton Mouton effect, optical ESR, optical forward backward birefringence, and a reinterpretation of antisymmetric light scattering and related phenomena in terms of $\mathbf{B}_{\Pi}$. It has also been deduced[18–21] that the quantum field equivalent of $\mathbf{B}_{\Pi}$ is the operator

$$\hat{B}_{\Pi} = B_0 \frac{\hat{J}}{\hbar} \tag{48}$$

where $\hat{J}$ is the quantized photon angular momentum, and $\hbar$ the reduced Planck constant. It has also been shown,[25] using the properties of the classical Lorentz transformation, that there can be no Faraday induction in free space due to a time derivative of the type $d\mathbf{B}_{\Pi}/dt$, produced, for example, by modulating a laser beam. (Note however, that Faraday induction occurs via the inverse Faraday effect[28] when a circularly polarized laser interacts with matter inside an induction coil.) The reason for this is that the Lorentz transformations do not allow free space $X$ and $Y$ components either of $\mathbf{B}_{\Pi}$ or of $\mathbf{E}_{\Pi}$, and also show that the $Z$ components $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ must be relativistically invariant.[25]

One of the simplest consequences of the presence of $\mathbf{B}_{\Pi}$ is an optical Zeeman effect, whose semiclassical theory regards $\mathbf{B}_{\Pi}$ as a classical vector.[24] In this approximation the theory of the optical Zeeman effect is the same as that of the conventional Zeeman effect,[29] with the conventional magnetostatic $\mathbf{B}_s$ replaced by $\mathbf{B}_{\Pi}$. In the simplest case, the Zeeman shift is proportional to

$$\Delta f = \hat{m} \cdot \frac{\mathbf{B}_{\Pi}}{h} \tag{49}$$

and therefore to the square root of the laser intensity $I_0^{1/2}$. This occurs in addition to an optical Zeeman shift caused[30] by the interaction of $\mathbf{E} \times \mathbf{E}^*$ with $\hat{\boldsymbol{\alpha}}''$, a mechanism that is proportional to intensity $I_0$. There appear to be no experimental investigations to date of the optical Zeeman effect, which requires only a minor modification of optical Stark effect apparatus to circularly polarize the pump laser.

Because $\mathbf{E}_{\Pi}$ is imaginary when $\mathbf{B}_{\Pi}$ is real, no simple physical effects are expected due to $\mathbf{E}_{\Pi}$, and significantly, none appears to have been reported in the literature.

The experimental evidence for the presence of uniform and time-independent components in the free space solutions of Maxwell's equations is available in at least three forms: (1) the inverse Faraday effect (IFE),[28] the optical Faraday effect (OFE),[30] and optically induced frequency shifts in NMR (ONMR).[27] In the IFE, bulk magnetization has been observed when a circularly polarized giant ruby laser pulse was passed through a sample in an inductance coil, thereby producing a measurable voltage that was not present in linear polarization and that changed sign with the sense of circular polarization. These are characteristic properties of $\mathbf{B}_{\Pi}$. In ONMR, a continuous wave argon ion laser was used[27] to shift NMR resonances and the shifts were much larger in circular than in linear polarization of the laser, too large to be explained by mechanisms based on the oscillating **E** and **B**, which time average to zero. For example it may be conjectured that the oscillating $\mathbf{B}(\mathbf{r}, t)$ induces in semiclassical theory a magnetic dipole moment in the electrons of a molecule in ONMR, a dipole moment that sets up a magnetic field at the resonating nucleus. However, this induced magnetic field would produce shifts much smaller than those observed,[27] and would time average to zero and no ONMR shift would be observable. The novel field $\mathbf{B}_{\Pi}$ does not time average to zero, and in principle sets up an interaction Hamiltonian $\hat{\mathbf{m}}_N \cdot \mathrm{Im}(\mathbf{B}_{\Pi})$, where $\hat{\mathbf{m}}_N$ is the nuclear magnetic dipole moment, causing ONMR shifts in circular polarization, as observed,[27] but not in linear polarization. In general, $\hat{\mathbf{B}}_{\Pi}$ is an operator in quantum field theory, and its interaction with the operator $\hat{\mathbf{m}}_N$ must be described properly in terms of quantization both in the applied laser field and in the nucleus. ONMR provides information about the nature of the interaction between $\hat{\mathbf{B}}_{\Pi}$ and the nucleus. However, there are several competing mechanisms in ONMR, and the data cannot yet be interpreted unequivocally. Recently, Frey et al.[30] have reported the optical Faraday effect in magnetic semiconductors, in which the polarization direction and phase of a laser beam are modified by interaction with the material. The authors interpret these changes in terms of nonlinear Faraday processes, but it is interesting to note that a plot of their experimentally observed rotation of the polarization of the laser by the sample against the square root of the laser intensity is a straight line within the experimental uncertainty (Fig. 1). This is the result expected from a mechanism of self-rotation based on the presence of the vector $\mathbf{B}_{\Pi}$ in the laser beam. These results are suggestive, but not unequivocal, because the straight line does not go through the origin in Fig. 1, and because it is not clear from the experimental arrangement of Frey et al.[30] whether the laser they used was circularly polarized. The output beam from the sample was analyzed by these authors with a Soleil compensator to determine its state of polarization, and the fact that a rotation of polarization was observed[30] (Fig. 1) suggests an excess of circular polarization in the output beam. A more critical test for the presence of $\mathbf{B}_{\Pi}$ would consist of a completely circularly polarized pump laser incident on a magnetic semiconductor (showing a giant Zeeman effect) together with a linearly polarized probe laser. The plane of polarization of the latter would be rotated by the $\mathbf{B}_{\Pi}$ vector of the pump, and this rotation should be proportional to the square root of the pump laser's intensity if there are no competing mechanisms.



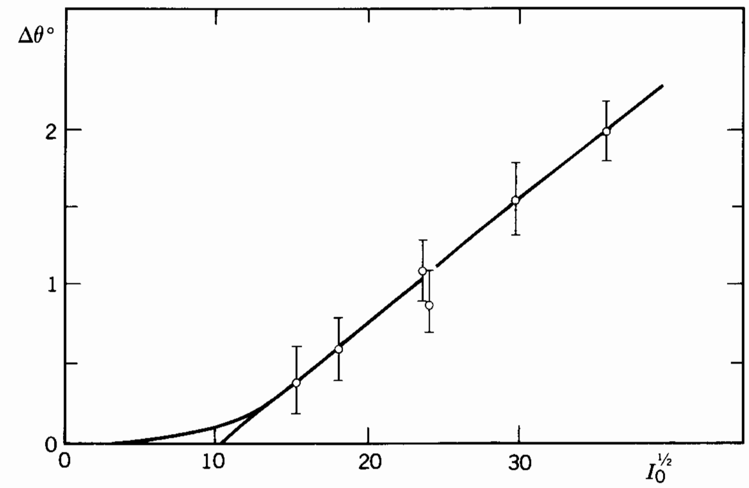


**Figure 1.** Plot of data from Ref. 30 of the optical Faraday effect, angle of rotation versu the square root of laser intensity. The points and uncertainty bars are those of Ref. 30. — Best fit line of Ref. 30; - - -, linear extrapolation (this work).

The sense of rotation should be reversed on reversing the sense of circula polarization of the pump, and should vanish when the pump is linearl polarized. The semiclassical theory of this effect is the same as that[31] o the conventional Faraday effect of 1846, with the magnetostatic field of th latter replaced by $\mathbf{B}_{\Pi}$ of the pump laser. The quantum field theory woul treat $\hat{\mathbf{B}}_{\Pi}$ as a quantum operator, Eq. (48), and there is no reason t suppose that the results in quantum field theory would be the same as tha in semiclassical theory, i.e., there are nonclassical effects, in general, du to $\mathbf{B}_{\Pi}$ treated as quantum field operators.

The available experimental evidence for the existence of $\mathbf{B}_{\Pi}$ is sugges tive, but not unequivocal; therefore, the challenge is to separate th particular influence of $\mathbf{B}_{\Pi}$ from the simultaneous influence of $\mathbf{E}(\mathbf{r}, t)$ an $\mathbf{B}(\mathbf{r}, t)$. It appears that one of the clearest ways of demonstrating th existence of $\mathbf{B}_{\Pi}$ would be through its characteristic square root intensit dependence, and through the fact that both vectors change sign with th sense of circular polarization, vanishing in linear polarization. If n experimental evidence for $\mathbf{B}_{\Pi}$ were found, such an eventuality would i itself be a major challenge to contemporary understanding of the nature o light and electromagnetic radiation in general. The reason for this is tha the vector $\mathbf{B}_{\Pi}$ is directly proportional (Eq. (13)) to the conjugate produc $\mathbf{E} \times \mathbf{E}^*$, in free space, and if the notion of conjugate product is accepted as is the contemporary practice, then $\mathbf{B}_{\Pi}$ must be accepted, and vice versa

If $\mathbf{B}_\Pi$ is real then $E_\Pi$ is imaginary, through Eq. (37), and it has been demonstrated in this work that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ do not affect the law of conservation of electromagnetic energy, the widely accepted continuity equation (8) of the classical theory of fields. The notions of $\mathbf{E} \times \mathbf{E}^*$, $\mathbf{B}_\Pi$, and $\mathbf{E}_\Pi$ are inextricably and ineluctably interrelated, therefore, and experimental evidence for the presence of any one is evidence for all. Conversely, if there is no apparent evidence for one, then all must not exist. The inverse Faraday effect has been interpreted through the notion of $\mathbf{E} \times \mathbf{E}^*$ (Refs. 7 and 8), but this provides an explanation in terms of only one mechanism, proportional to intensity. It has been argued here that there must be another mechanism present, proportional to the square root of intensity (the $\mathbf{B}_\Pi$ mechanism). If these are found experimentally, contemporary understanding would be strengthened. But if evidence for one mechanism (e.g., $\mathbf{E} \times \mathbf{E}^*$) is found and evidence for another (e.g., $\mathbf{B}_\Pi$) is not found, then the theory of electromagnetic fields would be challenged at the most fundamental level.

Clearly, the notion of $\mathbf{E} \times \mathbf{E}^*$ implies that this object is transmitted through free space in an electromagnetic plane wave, and when this wave meets particulate matter, an interaction Hamiltonian is formed between $\mathbf{E} \times \mathbf{E}^*$ and a material property. In atoms and molecules with net electronic angular momentum, this property is the vectorial polarizability vector $\hat{\boldsymbol{\alpha}}''$, well defined and accepted in semiclassical time-dependent perturbation theory, based on the time-dependent Schrödinger equation.[29] Since $\mathbf{B}_\Pi$ is directly proportional to $\mathbf{E} \times \mathbf{E}^*$, it cannot be argued that $\mathbf{E} \times \mathbf{E}^*$ exists and that $\mathbf{B}_\Pi$ does not. The source of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ is clearly the same as that of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. Furthermore, it has been shown that $\mathbf{B}_\Pi$ is part of the general solution of the equations of Maxwell, and is therefore phenomenologically indistinguishable from uniform, magnetostatic flux density, whose symmetry and units it possesses. It cannot therefore be argued that $\mathbf{B}_\Pi$ cannot form an interaction Hamiltonian with the appropriate material property (a magnetic dipole moment), and it has been shown in Eq. (19) that if $i\hat{\boldsymbol{\alpha}}'' \cdot \mathbf{E} \times \mathbf{E}^*$ is accepted as an interaction energy, then $\hat{\mathbf{m}} \cdot \mathrm{Im}(\mathbf{B}_\Pi)$ must also be accepted. Finally, the classical presence of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ must also have meaning in quantum field theory, where these vector fields become operators (Appendix B).

## APPENDIXA: LORENTZ COVARIANCE

The theory of the Lorentz covariance of Maxwell's equations show[10] that the complex quantity

$$\mathbf{F}_\Pi = \mathbf{E}_\Pi + ic\mathbf{B}_\Pi \tag{A.1}$$



is an invariant of the Lorentz transformation of special relativity. Therefore,

$$F_\Pi^2 = E_\Pi^2 - c^2 B_\Pi^2 + 2ic\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi \tag{A.2}$$

is an invariant of $\mathbf{F}_\Pi$ with respect to rotation in $(\mathbf{Z}, t)$ in Minkowski four space. This is equivalent to a rotation in the $X, Y$ plane through an imaginary angle in three dimensions. Thus, $E_\Pi^2 - c^2 B_\Pi^2$ and $\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi$ are the only two independent invariants of the antisymmetric four tensor of the electromagnetic field in the four-dimensional representation. This is the tensor $F^{ik}$, with

$$\begin{aligned} F_{ik}F^{ik} &= \text{inv.} \\ e^{iklm}F_{ik}F_{lm} &= \text{inv.} \end{aligned} \tag{A.3}$$

and where $e^{iklm}$ denotes the completely antisymmetric unit tensor of rank four.

It is important to note that these invariants are zero only when

$$E_\Pi^2 = c^2 B_\Pi^2 \tag{A.4}$$

and

$$\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi = 0 \tag{A.5}$$

Because, for complex $\mathbf{B}_\Pi$ and $\mathbf{E}_\Pi$,

$$|\mathbf{E}_\Pi| = \sqrt{2}E_0 \quad \textit{and} \quad |\mathbf{B}_\Pi| = \sqrt{2}E_0 \qquad E_0 = cB_0,$$

we obtain

$$E_\Pi^2 - c^2 B_\Pi^2 = 0 \tag{A.6}$$

However, because $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are parallel in $Z$, the propagation axis,

$$\mathbf{E}_\Pi \cdot \mathbf{B}_\Pi \neq 0 \tag{A.7}$$

If $F_\Pi^2 \neq 0$, it is known that $\mathbf{F}_\Pi = a\mathbf{n}$ (Ref. 10), where $\mathbf{n}$ is a complex unit vector ($nn^* = 1$). Using a complex rotation it is always possible to direct $\mathbf{n}$ along a coordinate axis,[10] and $\mathbf{n}$ becomes real, determining the directions of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$:

$$\mathbf{F}_\Pi = (E_\Pi + icB_\Pi)\mathbf{n} \tag{A.8}$$

so $\mathbf{E}_\Pi$ is parallel to $\mathbf{B}_\Pi$. Therefore, in any frame of reference, $\mathbf{E}_\Pi$ must be parallel to $\mathbf{B}_\Pi$; and there can be no components of either $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ perpendicular to the direction of the plane wave.

It is concluded that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ form a nonzero invariant, Eq. (A.7), of the Lorentz transformation of special relativity. The oscillating components $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ form zero invariants, because they are mutually perpendicular and $E_0 = cB_0$.

Furthermore, the dual transformation of special relativity corresponds to[10]

$$\begin{aligned} c\mathbf{B}^G &\rightarrow -\mathrm{i}\mathbf{E}^G \\ -\mathrm{i}\mathbf{E}^G &\rightarrow c\mathbf{B}^G \end{aligned} \tag{A.9}$$

which leaves the Maxwell equations invariant in vacuo. The dual transformation therefore corresponds to

$$\begin{aligned} c(\mathbf{B}(\mathbf{r},t) + \mathrm{i}\mathbf{B}_\Pi) &\rightarrow -\mathrm{i}\mathbf{E}(\mathbf{r},t) + \mathbf{E}_\Pi \\ -\mathrm{i}(\mathbf{E}(\mathbf{r},t) + \mathbf{E}_\Pi) &\rightarrow c(\mathbf{B}(\mathbf{r},t) + \mathbf{B}_\Pi) \end{aligned} \tag{A.10}$$

Therefore, the Maxwell equations are invariant in vacuo to the transformations $c\mathbf{B}_\Pi \rightarrow -\mathrm{i}\mathbf{E}_\Pi$ and $-\mathrm{i}\mathbf{E}_\Pi \rightarrow c\mathbf{B}_\Pi$. Exchanging $c\mathbf{B}_\Pi$ and $-\mathrm{i}\mathbf{E}_\Pi$ everywhere, or vice versa, the Maxwell equations are the same in any frame of reference, which is consistent with the fact that $\mathbf{E}_\Pi$ is parallel to $\mathbf{B}_\Pi$ in $Z$, and if $\mathbf{B}_\Pi$ is real, then $\mathbf{E}_\Pi$ is imaginary. A gauge transformation leaves the Lorentz relation unchanged, and leaves $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ unaltered. Thus, gauge invariance means that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are unaltered in any valid gauge.

It is important to note that the invariant in Eq. (A.1) is a complex quantity, and that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ also appear in Eqs. (4) and (5) as complex. Otherwise, the invariant in Eq. (A.6) would not vanish, and $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ would contribute to the electromagnetic energy density and flux density. This is essentially the result of the special theory of relativity, which uses four-dimensional Minkowski space:

$$m = (X, Y, Z, \mathrm{i}ct)$$

The Lorentz transformation describes rotations in this four space, and it follows that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are unaffected by rotations in spacetime. The fourth coordinate of Minkowski space is $\mathrm{i}ct$, i.e., imaginary. The reason is that this gives a space in which the four-dimensional Pythagorean theorem has the same form as the usual three-dimensional theorem. This is taken as a fundamental criterion of a Cartesian system. Maxwell's equations can be written in tensor form in Minkowski space, leading to the Lorentz transformation.

It is concluded that $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are entirely consistent with the Lorentz covariance of the Maxwell equations, and form invariants of the Lorentz transformation. The fields $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ are therefore physically meaningful in the classical theory of electromagnetic radiation. The free space vector

$$\mathbf{F}_\Pi = \mathbf{E}_\Pi + \mathrm{i}c\mathbf{B}_\Pi \tag{A.11}$$

is a nonzero invariant of the Lorentz transformation in the special theory of relativity. The existence of $\mathbf{B}_\Pi$ implies that Maxwell's equations in free space support the nonlinear solution:

$$\mathbf{B}^G(\mathbf{r},t) = \mathbf{B}(\mathbf{r},t) + \frac{1}{\mathrm{i}E_0 c}\mathbf{E}(\mathbf{r},t) \times \mathbf{E}^*(\mathbf{r},t) \tag{A.12}$$

Finally, the dual transformations show that the Maxwell equations are invariant to

$$\begin{aligned} E_0(\mathbf{i} + \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} &\rightarrow B_0(\mathbf{i} + \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} \\ B_0(\mathbf{j} - \mathrm{i}\mathbf{i})e^{\mathrm{i}\phi} &\rightarrow E_0(\mathbf{j} - \mathrm{i}\mathbf{i})e^{\mathrm{i}\phi} \\ -E_0\mathbf{k} &\rightarrow \mathrm{i}E_0\mathbf{k} \\ \mathrm{i}B_0\mathbf{k} &\rightarrow B_0\mathbf{k} \end{aligned} \tag{A.13}$$

so Eq. (40) is also satisfied by

$$\begin{aligned} \mathbf{B}^G &= \mathbf{B} - \mathrm{i}B_0\mathbf{k} \\ \mathbf{E}^G &= \mathbf{E} - E_0\mathbf{k} \end{aligned} \tag{A.14}$$

Therefore (A.14) is also a valid solution of Maxwell's equations, and it is therefore possible to obtain valid solutions of Maxwell's equations in which $\mathbf{E}_\Pi$ is real and $\mathbf{B}_\Pi$ is imaginary, or in which $\mathbf{E}_\Pi$ is imaginary and $\mathbf{B}_\Pi$ is real. Since $\mathbf{B}_\Pi$ from eqn. (18) is real, $\mathbf{E}_\Pi$ is imaginary.

It may also be verified that solutions of the type

$$\begin{aligned} \mathbf{E}^G &= \mathbf{E}(\mathbf{r},t) \pm E_0(\mathrm{i} - 1)\mathbf{k} \\ \mathbf{B}^G &= \mathbf{B}(\mathbf{r},t) \pm B_0(\mathrm{i} + 1)\mathbf{k} \end{aligned} \tag{A.15}$$

satisfy Maxwell's equations in vacuo and also relation (40) of the text. The most general solution of Maxwell's equations is therefore in this case:

$$\mathbf{E}^{G} = \frac{E_0}{\sqrt{2}}\left[(\mathbf{i} \pm \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} \pm \sqrt{2}\,(\mathrm{i}-1)\mathbf{k}\right]$$
$$\mathbf{B}^{G} = \frac{B_0}{\sqrt{2}}\left[(\mathbf{j} \mp \mathrm{i}\mathbf{i})e^{\mathrm{i}\phi} \pm \sqrt{2}\,(\mathrm{i}+1)\mathbf{k}\right] \tag{A.16}$$

where the normalisation factor $1/\sqrt{2}$ has been used, as is the standard practice.

Solutions (A.16) show that, in general, Maxwell's equations in vacuo support components in **i**, **j**, and **k** unit vectors in $X$, $Y$, and $Z$, respectively. In this case, the fields $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ are both complex:

$$\mathbf{B}_{\Pi} = B_0(1+\mathrm{i})\mathbf{k}$$
$$\mathbf{E}_{\Pi} = E_0(-1+\mathrm{i})\mathbf{k} \tag{A.17}$$

## APPENDIX B: QUANTIZATION OF $\mathbf{B}_{\Pi}$ AND $\mathbf{E}_{\Pi}$

Defining the annihilation and creation operators

$$\hat{a}(t) = \hat{a}(0)e^{-\mathrm{i}\omega t}$$
$$\hat{a}^{+}(t) = \hat{a}^{+}(0)e^{\mathrm{i}\omega t} \tag{B.1}$$

the quantized equivalent of the oscillating electric field

$$\mathbf{E}(t) = \frac{E_0}{\sqrt{2}}e^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}}(\mathbf{i}e^{-\mathrm{i}\omega t} + \mathrm{i}\mathbf{j}e^{-\mathrm{i}\omega t}) \tag{B.2}$$

becomes the operator

$$\hat{\mathbf{E}}(t) = \frac{E_0}{\sqrt{2}}e^{\mathrm{i}\boldsymbol{\kappa}\cdot r}(\hat{a}_X(t)\mathbf{i} + \mathrm{i}\hat{\mathbf{a}}_Y(t)\mathbf{j}) \tag{B.3}$$

Similarly,

$$\hat{\mathbf{E}}^{*}(t) = \frac{E_0}{\sqrt{2}}e^{-\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}}(\hat{a}_X^{+}(t)\mathbf{i} - \mathrm{i}\hat{a}_Y^{+}(t)\mathbf{j}) \tag{B.4}$$



and

$$\mathbf{E}\times\mathbf{E}^{*} = -\mathrm{i}\frac{E_0^2}{2}\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right)\mathbf{k} \tag{B.5}$$

where

$$\hat{a}_X\hat{a}_Y^{+} \equiv \hat{a}_X(t)\hat{a}_Y^{+}(t) = \hat{a}_X(0)\hat{a}_Y^{+}(0) \tag{B.6}$$

and

$$E_0^2 = \frac{2\hbar\omega}{\varepsilon_0 L_0^3}$$

Here $L_0^3$ is the volume of quantization[9] and $\varepsilon_0$ the vacuum permittivity.

The expectation value of the operator in Eq. (B.5) is the classical conjugate product $\mathbf{E}\times\mathbf{E}^{*}$:

$$\langle n|\hat{\mathbf{E}}\times\hat{\mathbf{E}}^{*}|n\rangle = -\mathrm{i}E_0^2\mathbf{k} \tag{B.7}$$

implying that

$$\langle n|\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}|n\rangle = 2 \tag{B.8}$$

It follows from Eq. (B.5) that

$$\hat{B}_{\Pi} = \tfrac{1}{2}B_0\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right)\mathbf{k} \tag{B.9}$$

where

$$B_0 = \left(\frac{2\mu_0\omega}{L_0^3}\right)^{1/2} \tag{B.10}$$

with $\mu_0$ denoting the vacuum permeability. As is the standard practice in quantum field theory, $|n\rangle$ is an eigenfunction whose eigenvalues are nonnegative integers $n$, i.e.,

$$\hat{N}|n\rangle = n|n\rangle \tag{B.11}$$

where $\hat{N}$ is the photon number operator, whose eigenstates are photon number states of the quantized field. The photon is a quantum of the field, with energy $\hbar\omega$. Thus, $\hat{a}^{+}$ operates on $|n\rangle$ to produce a field increment of energy $\hbar\omega$, i.e., to produce a photon, and $\hat{a}$ operates conversely:

$$\begin{aligned}\hat{a}^{+}|n\rangle &= (n+1)^{1/2}|n+1\rangle \\ \hat{a}^{-}|n\rangle &= n^{1/2}|n-1\rangle\end{aligned} \tag{B.12}$$

the normalization factor being chosen so that[9]

$$\langle n|n'\rangle = \delta_{nn'} \tag{B.13}$$

which is the orthonormality condition.

It is instructive to note that Eq. (B.9) for the operator $\hat{B}_\Pi$ can be derived independently of the conjugate operator $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$, showing, inter alia, that $\hat{\mathbf{B}}_\Pi$ *and* $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ are rigorously proportional to each other in quantum field theory. The independent derivation proceeds from a direct quantization of the classical magnetostatic field

$$\mathbf{B}_\Pi = B_0\mathbf{k} \tag{B.14}$$

where we have made no assumptions concerning the origin of this field. Here $\mathbf{k}$ is a axial unit vector in the $Z$ axis of the frame $(X, Y, Z)$:

$$\mathbf{k} = \mathbf{i} \times \mathbf{j} \tag{B.15}$$

where $\mathbf{i}$ is a polar unit vector in $X$ and $\mathbf{j}$ a polar unit vector in $Y$. Thus, $B_0$ is a scalar magnetic flux density amplitude (tesla).

It is always possible to write Eq. (B.14) as

$$\begin{aligned}\mathbf{B}_\Pi &= B_0\mathbf{k} \equiv \tfrac{1}{2}B_0 e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} \\ &\quad \times (e^{-i\omega t}\mathbf{i} \times e^{i\omega t}\mathbf{j} - e^{-i\omega t}\mathbf{j} \times e^{i\omega t}\mathbf{i}) \\ &= \tfrac{1}{2}B_0 e^{-i\omega t} e^{i\omega t} \\ &\quad \times \left( \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \right)\end{aligned} \tag{B.16}$$

a purely mathematical identity, which leads to the well-known conclusion of tensor algebra that an axial vector is equivalent to a second-rank polar tensor.

If we now assume that $\omega$ is the angular frequency of the classical electromagnetic plane wave, then the quantized form of any magnetic field operator of the form (B.14) carried by such a wave pattern in quantum



field theory must be

$$\hat{\mathbf{B}}_\Pi = \frac{B_0}{2}\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right)\mathbf{k} \tag{B.17}$$

which, except for sign, is the same as Eq. (B.9). The sign of $\hat{\mathbf{B}}_\Pi$ is switched[1–7] by switching from left to right circular polarization, and in consequence $\hat{\mathbf{B}}_\Pi$ can always be defined as plus or minus. Equation (B.17) has been derived directly from Eq. (B.16) using the definitions (B.1), and using no other assumption. This shows that the magnetic field operator $\hat{\mathbf{B}}_\Pi$ and the conjugate product operator $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ are rigorously proportional in quantum field theory. Both are described by the operator $(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+})$ whose expectation values between states is always 2.

Further insight to the physical interpretation of $\hat{\mathbf{B}}_\Pi$ *and* $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ can be gained by using the quantum field definition[9] of the Stokes parameter $S_3$:

$$\hat{S}_3 = -\frac{E_0^2}{2}\left(\hat{a}_X\hat{a}_Y^{+} - \hat{a}_Y\hat{a}_X^{+}\right) \tag{B.18}$$

so that

$$\hat{\mathbf{B}}_\Pi = -\frac{\hat{S}_3}{E_0 c}\mathbf{k} \tag{B.19}$$

showing that $\hat{\mathbf{B}}_\Pi$ *and* $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^*$ are both directly proportional to the scalar operator $\hat{S}_3$, the third Stokes operator of quantum field theory.

Furthermore, the Stokes operators $\hat{S}_1$, $\hat{S}_2$, *and* $\hat{S}_3$ obey the commutator equations of angular momentum in quantum field theory,[9] showing that $\hat{\mathbf{B}}_\Pi$ has the properties of quantized angular momentum of the electromagnetic wave. Standard theory[9] shows that

$$\langle n|\hat{S}_3|n\rangle = \langle\hat{\sigma}_Z\rangle = \frac{\langle\hat{J}_Z\rangle}{\hbar} \tag{B.20}$$

where $\hat{\sigma}_Z$ is the Pauli matrix operator:

$$\hat{\sigma}_Z \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.21}$$

so that

$$\boldsymbol{\psi} = \frac{E_0}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \tag{B.22}$$

is a spinor. The operator $\hat{J}_Z$ is defined by

$$\langle n|\hat{J}_Z|n\rangle = \hbar S_3 \tag{B.23}$$

Using Eq. (B.21), we obtain

$$\hat{\mathbf{B}}_{\Pi} = B_0 \frac{\hat{J}_Z}{\hbar}\mathbf{k} = B_0 \frac{\hat{J}}{\hbar} \tag{B.24}$$

with $B_0$ given by Eq. (B.10) in SI units.

It is seen from Eq. (B.23) that the expectation value of $\hat{J}_Z$ between eigenfunctions $|n\rangle$ is $S_3$, the classical third Stokes parameter, multiplied by $\hbar$, the unit angular momentum in quantum theory. The expectation value of $\hat{J}_Z$ is $S_3$ units of quantized angular momentum for any eigenstate $|n\rangle$ of the quantized field. Classically, the third Stokes operator is

$$S_3 = -\frac{\mathrm{i}}{2}(E_X \mathrm{i} E_Y^* + \mathrm{i} E_Y E_X^*) = \tfrac{1}{2}(E_X E_Y^* + E_Y E_X^*) \equiv \mathrm{E}_{\mathrm{L}}^2 \tag{B.25}$$

a real quantity. The intensity of the beam is classically

$$I_0 = \varepsilon_0 c E_{\mathrm{L}}^2 \tag{B.26}$$

showing that the expectation value of the angular momentum operator $\hat{\mathbf{J}}$ is

$$\frac{1}{\hbar}\langle n|\hat{\mathbf{J}}|n\rangle = \frac{I_{\mathrm{L}}}{\varepsilon_0 c} \tag{B.27}$$

which is directly proportional to the intensity of a beam that is fully left circularly polarized. In a beam that has elements of both left and right circular polarization,

$$\frac{1}{\hbar}\langle n|\hat{\mathbf{J}}|n\rangle = \frac{1}{\varepsilon_0 c}(I_{\mathrm{L}} - I_{\mathrm{R}}) \tag{B.28}$$

If the beam were to consist of a wave pattern corresponding to one photon of energy $\hbar\omega$, then $|n\rangle = |1\rangle$ and $\langle 1|\hat{\mathbf{J}}|1\rangle$ is the expectation value of $\hat{\mathbf{J}}$ for one photon. In this case, $\hat{\mathbf{B}}_{\Pi}$ is the magnetic flux density operator of one photon, whose scalar magnitude we denote $B_0^{(1)}$:

$$B_0^{(1)} = \left(2\mu_0 \frac{\hbar\omega}{L_0^3}\right)^{1/2} \tag{B.29}$$



which is proportional to the square root of $\hbar\omega$, the energy of the single photon under consideration. The energy of $n$ photons is describable by the expectation value[9]

$$\langle n|\hat{H}^{\mathrm{R}}|n\rangle = \left(n + \tfrac{1}{2}\right)\hbar\omega \tag{B.30}$$

i.e., by an integer number $n$ of energy quanta $\hbar\omega$, plus a "background" $\hbar\omega/2$ independent of $n$. Therefore,

$$nB_0^{(1)2} = \frac{2n\omega_0}{L_0^3}\hbar\omega \tag{B.31}$$

and the energy $n\hbar\omega$ is proportional to $nB_0^{(1)2}$. Using $E_0 = cB_0$ and Eq. (B.26), it becomes clear that the intensity of a beam of $n$ photons is proportional to $nB_0^{(1)2}$, so that $B_0^{(1)}$ is an elementary quantum of magnetostatic flux density associated with one photon. This is in analogy with the fact that the quantum of energy associated with the photon is $\hbar\omega$. From Eq. (B.29), $B_0^{(1)}$ vanishes if $\omega = 0$, i.e., if the frequency of the wave is zero. In this case the energy $\hbar\omega$ is zero, and there are no photons. Alternatively, if $L_0^3 \to \infty$, i.e., if the quantization volume tends to infinity, then $B_0^{(1)}$ tends to zero, even for finite $\omega$.

Under all other conditions, $B_0^{(1)}$ is nonzero, and produces finite and measurable physical effects, as described in the text.

The quantization of the imaginary $\mathbf{E}_{\Pi}$ proceeds similarly using the classical dual transformation

$$\mathbf{B}_{\Pi} \to -\frac{\mathrm{i}}{c}\mathbf{E}_{\Pi} \tag{B.32}$$

And Eq. (B.9)

$$\mathrm{i}\hat{\mathbf{E}}_{\Pi} = \tfrac{1}{2}E_0 \mathrm{i}\left(\hat{a}_X \hat{a}_Y^+ - \hat{a}_Y \hat{a}_X^+\right)\mathbf{k} \tag{B.33}$$

whose expectation value is

$$\mathbf{E}_{\Pi} = \langle n|\hat{\mathbf{E}}_{\Pi}|n\rangle = -E_0\mathbf{k} \tag{B.34}$$

is an acceptable definition (see Appendix A) of $\mathbf{E}_{\Pi}$. Here $\mathbf{k}$ is of course a polar unit vector.

It is seen that both $\hat{\mathbf{B}}_{\Pi}$ *and* $\mathrm{i}\hat{\mathbf{E}}_{\Pi}$ are defined in terms of the operator $\hat{a}_X \hat{a}_Y^+ - \hat{a}_Y \hat{a}_X^+$, which operates on any number state $|n\rangle$ to give the constant expectation value of 2. This expectation value is independent of the number state $|n\rangle$ of the photons, and generates the third Stokes parameter $S_3$ of the classical field.

Classically, the $\mathbf{B}_\Pi$ is an axial vector, $\hat{P}$-positive and $\hat{T}$-negative, and proportional to angular momentum. The field $\mathbf{E}_\Pi$ is a polar vector, $\hat{P}$-negative and $\hat{T}$-positive, and cannot therefore be proportional to an angular momentum. It is essential to note, therefore, that $\mathbf{k}$ in Eq. (B.9) is an axial unit vector ($\hat{T}$-negative and $\hat{P}$-positive) and that $\mathbf{k}$ in Eq. (B.33) is a polar unit vector ($\hat{T}$-positive and $\hat{P}$-negative).

Finally, we note that $\hat{a}_X\hat{a}_Y^* - \hat{a}_Y\hat{a}_X^*$ operates to give an eigenvalue of 2, and this does not change the energy $n\hbar\omega$ of $n$ photons. This is in agreement with the classical theory (see text), which shows that $\mathbf{B}_\Pi$ and $\mathbf{E}_\Pi$ do not contribute to the field energy.

## APPENDIX C: DEFINITION OF $\mathbf{B}_\Pi$ AND $\mathbf{E}_\Pi$ IN TERMS OF THE VECTOR POTENTIAL IN FREE SPACE

In free space, the oscillating fields **E** and **B** of the plane wave are defined in terms of the vector potential **A**. Using the Coulomb gauge:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi; \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{C.1}$$

In free space the scalar part, $\nabla\phi$ is zero and the Lorentz condition and Coulomb gauge are both describable by

$$\nabla \cdot \mathbf{A} = 0 \tag{C.2}$$

From the definition in the text,

$$\mathbf{B}_\Pi = \frac{1}{E_0 c}\left(\frac{\partial \mathbf{A}}{\partial t} \times \frac{\partial \mathbf{A}^*}{\partial t}\right) \equiv B_0\mathbf{k} \tag{C.3}$$

where

$$\mathbf{E}^* = -\frac{\partial \mathbf{A}^*}{\partial t} - \nabla\phi^* \tag{C.4}$$

Using the condition for conservation of energy,

$$\mathbf{E}_\Pi \times \mathbf{B} = \mathbf{B}_\Pi \times \mathbf{E} \tag{C.5}$$



with the definition (C.3) implies

$$\mathbf{E}_\Pi = \mathrm{i}E_0\mathbf{k} \tag{C.6}$$

and

$$\mathbf{B} \times \mathbf{E}_\Pi = \mathrm{i}B_0\mathbf{E} \tag{C.7}$$

From (C.1) in (C.7),

$$(\nabla \times \mathbf{A}) \times \mathbf{E}_\Pi = \mathrm{i}B_0\mathbf{E} = -\mathrm{i}B_0\left(\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) \tag{C.8}$$

which defines $\mathbf{E}_\Pi$ in terms of **A**.

Equation (C.8) can be simplified to

$$\nabla(\mathbf{E}_\Pi \cdot \mathbf{A}) = \mathrm{i}B_0\left(\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) \tag{C.9}$$

using

$$(\nabla \times \mathbf{A}) \times \mathbf{E}_\Pi = \mathbf{A}(\mathbf{E}_\Pi \cdot \nabla) - \nabla(\mathbf{E}_\Pi \cdot \mathbf{A}) \tag{C.10}$$

and

$$\mathbf{E}_\Pi \cdot \nabla = \nabla \cdot \mathbf{E}_\Pi = 0 \tag{C.11}$$

Note that (C.9) is a type of continuity equation which defines the imaginary $\mathbf{E}_\Pi$ in terms of **A** of the oscillating components **E** and **B** of the plane wave.

In texts on the electromagnetic plane wave it is usually asserted that **E** and **B** are transverse plane waves, with no components in the direction of propagation. Equations (C.1) and (C.2) are usually taken as justification for this conclusion. Most texts assert that electromagnetic plane waves in vacuo are necessarily time-varying, because the solutions for constant $E$ and $B$ from Maxwell's equations in the absence of charge and current are zero. While this is true for linear solutions, we can form a nonlinear solution, Eq. (C.3), for $\mathbf{B}_\Pi$, which is well defined as in this paper, and which is a product of time-varying solutions. With the condition (C.5), derived in the text of this paper, the field $\mathbf{E}_\Pi$ is also well defined in terms of **A** as in Eq. (C.9), a novel continuity equation.

Nonlinear solutions of Maxwell's equations therefore support the existence of $\mathbf{E}_\Pi$ and $\mathbf{B}_\Pi$ in the axis of propagation of the plane wave in vacuo.

## APPENDIX D: THE CONSTANT OF INTEGRATION IN EQUATION (35)

Most generally, from Eq. (35),

$$\mathbf{E}_\Pi \times \mathbf{B} = \mathbf{B}_\Pi \times \mathbf{E} + \text{constant} \tag{D.1}$$

The dual transformation of special relativity means that $\mathbf{E}_\Pi$ and $-(c/\mathrm{i})\mathbf{B}_\Pi$, for example, are indistinguishable solutions of Maxwell's equations; i.e., it is possible to replace $\mathbf{E}_\Pi$ everywhere by $-(c/\mathrm{i})\mathbf{B}_\Pi$ without changing the Maxwell equations, and therefore without changing the solutions to the equations. The dual transformation, however, does not affect the constant in Eq. (D.1), which is independent of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{E}_\Pi$, and $\mathbf{B}_\Pi$. Thus, applying the dual transform,

$$\mathbf{B}_\Pi \times \mathbf{E} = \mathbf{E}_\Pi \times \mathbf{B} + \text{constant} \tag{D.2}$$

Adding Eqs. (D.1) and (D.2) yields

$$\text{Constant} = 0$$

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# *ON LONGITUDINAL FREE SPACETIME ELECTRIC AND MAGNETIC FIELDS IN THE EINSTEIN-DE BROGLIE THEORY OF LIGHT*

## I. INTRODUCTION

It is usually concluded in electrodynamical literature[1–16] that the photon is massless and that the range of the electromagnetic field is infinite. This conclusion is not, however, supported by experimental data. To the contrary, Vigier[17] has recently reviewed a substantial amount of evidence that leads to the conclusion of finite photon rest mass. These data include, to take two of many examples, the direction-dependent anisotropy of the frequency of light in cosmology and frequently observed anomalous red shifts.