

DERIVATION OF THE VACUUM LONGITUDINAL FIELD $\mathbf{B}^{(3)}$ FROM THE DIRAC EQUATION OF THE ELECTRON IN THE ELECTROMAGNETIC FIELD**M. W. Evans***Department of Physics
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By solving the Dirac equation for the motion of an electron (e) in the circularly polarized electromagnetic field it is shown that the intrinsic electron spin forms an interaction Hamiltonian with a time independent field $\mathbf{B}^{(3)}$ of electromagnetic radiation in the vacuum. In the same way as intrinsic spin is a fundamental property of the electron, $\mathbf{B}^{(3)}$ is therefore a fundamental and intrinsic property of the vacuum electromagnetic field.

Key words: $\mathbf{B}^{(3)}$ field, Dirac equation.

1. INTRODUCTION

The recent emergence [1-3] of a novel, longitudinal magnetic field $\mathbf{B}^{(3)}$ of vacuum electromagnetism has led to detailed theoretical developments reported in Refs. [4-7]. The precise experimental conditions under which $\mathbf{B}^{(3)}$ can be observed have also been defined [8], using its characteristic $I_0^{1/2}$ dependence, where I_0 is the power density of the electromagnetic beam. In this Letter it is shown that $\mathbf{B}^{(3)}$ is a direct outcome of the Dirac equation for the quantum relativistic dynamics of an electron in the electromagnetic field, and is therefore a fundamental vacuum property of the field. The method of derivation used here is a simple adaptation of that [6] used to show the existence of intrinsic electron spin in an ordinary static magnetic field. The latter is

replaced by circularly polarized electromagnetic radiation of which $\mathbf{B}^{(3)}$ is an irremovable vacuum property. Section 2 sets up the Dirac equation for this problem, and Section 3 describes the method of solution. It is deduced that the intrinsic electron spin forms an interaction Hamiltonian with $\mathbf{B}^{(3)}$, and with $\mathbf{B}^{(3)}$ alone, whenever an electron is influenced by an electromagnetic field.

2. THE DIRAC EQUATION FOR THE MOTION OF AN ELECTRON IN THE ELECTROMAGNETIC FIELD

The electromagnetic field is represented by the four-potential A_μ and the Dirac equation of e in A_μ is obtained by replacing the electronic four-momentum [9] p_μ by the sum $p_\mu + eA_\mu$. This is the conventional *minimal prescription*, which is the result of gauge invariance in the usual $O(2)$ electromagnetic gauge geometry. It is shown in this Letter that this standard procedure demonstrates the existence of the novel field $\mathbf{B}^{(3)}$ in the vacuum. The Dirac equation is written, in S.I. units and Minkowski's notation,

$$\gamma_\mu(\mu_\mu + eA_\mu)\psi(\mathbf{p}) = -m_0c\psi(\mathbf{p}), \quad (1)$$

where γ_μ is the Dirac matrix [10] and $\psi(\mathbf{p})$ the Dirac four-spinor. In our notation,

$$\gamma_\mu := (\boldsymbol{\gamma}, i\gamma^{(0)}), \quad A_\mu := \left(\mathbf{A}, \frac{i\phi}{c}\right), \quad p_\mu := \left(\mathbf{p}, \frac{iEn}{c}\right), \quad (2)$$

and m_0 is the electron's mass. Equation (1) is therefore a relativistic equation of quantum mechanics. In the standard representation [10] it splits into

$$\begin{aligned} (En + e\phi)u - c\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A})v &= m_0c^2u, \\ -(En + e\phi)v + c\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A})u &= m_0c^2v, \end{aligned} \quad (3)$$

where $\boldsymbol{\sigma}$ denotes the Pauli matrices and the wave functions u and v are spinors. Standard methods of solution [10] give the following wave equation in the limit $e\phi \ll 2m_0c^2$:

$$\hat{W}u = Hu, \quad (4)$$

where

$$H = \frac{1}{2m_0} (\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}))^2 - e\phi. \quad (5)$$

Since H is a Hamiltonian it is invariant with time, and is the Hamiltonian for the quantum relativistic description of e in A_μ . It splits [10] into

$$H_{class} = \frac{1}{2m_0} (\mathbf{p} + e\mathbf{A}) \cdot (\mathbf{p} + e\mathbf{A}) - e\phi, \quad (6)$$

which has a classical counterpart, and the purely quantum mechanical

$$H_{spin} = \frac{i}{2m_0} \boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}) \times (\mathbf{p} + e\mathbf{A}), \quad (7)$$

where the cross product is interpreted in terms of operators, since it has no classical meaning. (Classically, the cross product of a vector with itself is zero.)

3. EMERGENCE OF $B^{(3)}$ FROM EQUATION (7)

H_{spin} from Eq. (7) is a time invariant description of the interaction of the quantized intrinsic spin of the electron with the electromagnetic field. The electron spin in this equation has no classical meaning. In H_{spin} , \mathbf{A} is however a classical vector potential and \mathbf{p} is henceforth denoted $\hat{\mathbf{p}}$ to distinguish it as an operator of quantum mechanics, $\hat{\mathbf{p}} := i\hbar\nabla$, where \hbar is the Dirac constant, an angular momentum magnitude. These definitions produce the commutators [10]

$$[\hat{p}_i, A_j] = \hat{p}_i A_j - A_j \hat{p}_i = -i\hbar \frac{\partial A_j}{\partial x_i}, \quad (8a)$$

$$[A_i, \hat{p}_j] = A_i \hat{p}_j - \hat{p}_j A_i = i\hbar \frac{\partial A_i}{\partial x_j}, \quad (8b)$$

i.e., $\hat{\mathbf{p}}$ operates on \mathbf{A} but \mathbf{A} is not an operator. Addition of (8a) and (8b) gives [10] the magnetic flux density

$$B_k = \frac{i}{\hbar}([\hat{P}_i, A_j] + [A_i, \hat{P}_j]), \quad (9)$$

which can be expressed as [10]

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{i}{\hbar}(\hat{\mathbf{P}} \times \mathbf{A} + \mathbf{A} \times \hat{\mathbf{P}}). \quad (10)$$

This reduces H_{spin} to the form

$$H_{spin} = \frac{e\hbar}{2m_0} \boldsymbol{\sigma} \cdot \mathbf{B}, \quad (11)$$

where $\hbar\boldsymbol{\sigma}$ is the electronic spin angular momentum. Since H_{spin} and $\hbar\boldsymbol{\sigma}$ are independent of time, so must be the electromagnetic field property \mathbf{B} . For electromagnetic radiation propagating in the Z (or (3)) axis, this identifies \mathbf{B} as $\mathbf{B}^{(3)}$, the novel spin field introduced in Refs [1-7]. Eq. (10) shows therefore that $\mathbf{B}^{(3)}$ can be defined in terms of a vector potential, as for any magnetic field, and this vector potential should produce the optical equivalent of the Aharonov-Bohm effect [5]. It is clear that the precise conditions under which this effect is observable must be estimated in relativistic quantum field theory, for example it may be preferable to use radiation at microwave frequencies (see Ref. [8]) rather than a visible frequency laser. This is an interesting prospect for further work. In Eq. (11), $\hbar\boldsymbol{\sigma}$ is inherently quantum mechanical in nature so the classical $\mathbf{B}^{(3)}$ is an expectation value of the quantized field. Our final result is therefore

$$H_{spin} = \frac{e\hbar}{2m_0} \boldsymbol{\sigma} \cdot \mathbf{B}^{(3)} \quad (12)$$

and is the quantum relativistic description of the interaction of the intrinsic electron spin with the electromagnetic field.

4. DISCUSSION

Equation (12) conclusively indicates the presence of $\mathbf{B}^{(3)}$ in the vacuum through its interaction with the intrinsic electronic spin $\hbar\boldsymbol{\sigma}$. The vacuum field $\mathbf{B}^{(3)}$ is a non-zero, time invariant magnetic flux density [1-7], and is an intrinsic property of the field in the same way that electronic

spin is an intrinsic property of the electron. The importance of this result cannot be over-emphasized, because $\mathbf{B}^{(3)}$ has no existence in our conventional view of $O(2)$ vacuum electrodynamics [11], yet emerges directly from the Dirac equation of e in A_μ . This finding has several fundamental consequences for field theory, which are developed in detail in a forthcoming monograph [6]. For example, the vacuum electromagnetic field has a longitudinal ((3)) component, indicating that the quantized photon is massive [6]. Another fundamentally important consequence is that the conventional $O(2)$ gauge geometry is replaced by a non-Abelian $O(3)$ gauge geometry [6] and the Maxwell equations thereby generalized [6]. Since $\hbar\sigma$ for the electron has been verified experimentally on countless occasions [10] the Hamiltonian H_{spin} is zero unless $\mathbf{B}^{(3)}$ is non-zero. The magnitude of H_{spin} is proportional to $I_0^{1/2}$, where I_0 is the power density of the vacuum electromagnetic field, and in principle, the characteristic $I_0^{1/2}$ dependence of $\mathbf{B}^{(3)}$ [8] is observable in optical effects such as Zeeman splitting due to H_{spin} . It is also observable by microwave magnetization of a plasma as discussed elsewhere [8].

It can be shown from considerations of H_{class} of Eq. (6) that $\mathbf{B}^{(3)}$ completely determines the spinning trajectory of e in A_μ . Consider, for simplicity, Eq. (6) with $\phi = 0$. If \mathbf{A} is the vector potential of $\mathbf{B}^{(3)}$, then

$$\mathbf{B}^{(3)} = \nabla \times \mathbf{A}, \quad \mathbf{A} = \frac{1}{2} \mathbf{B}^{(3)} \times \mathbf{r} = \frac{1}{2} B^{(3)} \mathbf{v}, \quad (13)$$

where $\mathbf{r} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$. Therefore $\mathbf{v} = -Y\mathbf{i} + X\mathbf{j}$ is a possible representation in Cartesian coordinates; and the term in \mathbf{A} in Eq. (13) becomes

$$\mathbf{H}^{(1)} := \frac{e}{m_0} \mathbf{A} \cdot \mathbf{p} = \frac{e}{2m_0} \mathbf{B}^{(3)} \times \mathbf{r} \cdot \mathbf{p} = \frac{e}{2m_0} \mathbf{B}^{(3)} \cdot \mathbf{r} \times \mathbf{p}. \quad (14)$$

In order for this to be non-zero, $\mathbf{r} \times \mathbf{p}$ must be in the same (3) direction as $\mathbf{B}^{(3)}$, and so defines the orbital angular momentum of e in A_μ ,

$$\mathbf{J}^{(3)} = \mathbf{r} \times \mathbf{p}. \quad (15)$$

So

$$H^{(1)} = \frac{e}{2m_0} \mathcal{J}^{(3)} \cdot \mathbf{B}^{(3)} = -\mathbf{m}^{(3)} \cdot \mathbf{B}^{(3)}, \quad (16)$$

where $\mathbf{m}^{(3)}$ is the magnetic dipole moment induced by the electromagnetic field. As shown elsewhere [5, 8], $\mathcal{J}^{(3)}$ can be defined by the relativistic but classical Hamilton-Jacobi equation of e in A_μ ,

$$\mathcal{J}^{(3)} = \frac{e^2 c^2}{\omega^2} \left(\frac{B^{(0)}}{(m_0^2 \omega^2 + e^2 B^{(0)2})^{\frac{1}{2}}} \right) \mathbf{B}^{(3)}, \quad (17)$$

Therefore, the magnetization by circularly polarized electromagnetic radiation of N non-interacting electrons in a plasma is

$$\mathbf{M}^{(3)} = N\mathbf{m}^{(3)} = -\frac{Ne}{2m_0} (\hbar\boldsymbol{\sigma} + \mathcal{J}^{(3)}). \quad (18)$$

This is the rigorous description of the inverse Faraday effect [11], completely dependent on the existence of $\mathbf{B}^{(3)}$ in the vacuum.

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