

Paper 2

On the Use of a Complex Vector Potential in the Minimal Prescription in the Dirac Equation

It is argued that the use of a complex vector potential in the minimal prescription maintains the basic Hermitian property of the Hamiltonian operator in Dirac's equation for the interaction of a fermion with the classical electromagnetic field. This is demonstrated by setting up the Dirac equation for a complex vector potential and for its complex conjugate, then forming a pure real Hamiltonian.

Key words: Minimal prescription; Dirac equation; complex vector potential.

2.1 Introduction

In the standard theory [1] the minimal prescription is used in the Dirac equation with a real vector potential. This method reproduces the standard description of the Stern-Gerlach experiment but does not allow for a coupling between the conjugate product [2—6] of the electromagnetic field

and the Pauli matrix of the fermion. In this paper we show that the use of a complex vector potential in the minimal prescription results in a pure real Hamiltonian operator which maintains the Hermitian properties of the Dirac equation intact *and* gives a direct coupling between the conjugate product and the Pauli matrix σ . In Sec. 2.2 we set up the equations necessary for this demonstration and solve them in Sec. 2.3. A short discussion is given of the consequences to nuclear magnetic resonance and electron spin resonance of the existence of such a direct coupling.

2.2 The Dirac Equation with a Complex Vector Potential

The use of a complex vector potential to describe electromagnetic radiation is a mathematical procedure in classical electrodynamics in which it is understood that the physical part is the real part [7,8]. In quantum mechanics the eigenvalues of Hermitian operators are real [1,9] and the eigenstates corresponding to different eigenvalues of Hermitian operators are orthogonal. Therefore it seems to be assumed implicitly that physical eigenvalues generated by the Schrödinger or Dirac equation must be pure real in order to be physical. This is a different rule from the one used in classical electrodynamics. In this paper we consistently use the same rule for both classical and quantum mechanics, and assume that the real part of a complex eigenvalue is physical. Therefore we use a complex, multiplicative, vector operator A multiplying the wavefunction (eigenfunction) in the basic wave equation, in this case a Dirac equation.

Proceeding on the basis of this working hypothesis, two Dirac equations are written for the interaction of the fermion with the classical electromagnetic field, represented respectively with a complex A and its conjugate A^* . The real parts of A and A^* are the same, so if we work on the basis of the rule that the real part of the complex operator is physical, the two Dirac equations become the same. Therefore the use of A and A^* in this way is a working mathematical hypothesis, one which leads to a pure real (and Hermitian) Hamiltonian operator. It is shown in Sec. 2.2 that this method leads to a well known result first given by Volkov [10] but in addition gives the interaction energy between the conjugate product $A \times A^*$

of the classical radiation field with the Pauli spinor σ . This interaction energy is proportional to intensity I divided by the square of angular frequency ω .

The Dirac equation is

$$H\psi = E\psi, \quad (2.2.1)$$

where the Hamiltonian operator is

$$H = c\alpha \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A} \right) + \beta mc^2 + eV. \quad (2.2.2)$$

Here ψ is the four-component Dirac spinor [1], and E the energy eigenvalue. The physical energy eigenvalue is real, so it is assumed usually that A must be real. In Eq. (2.2.2) we use Gaussian units [11]. Here c is the velocity of light, \mathbf{p} the assumed real momentum of the fermion, e its charge and m its mass. The scalar potential V is also assumed to be pure real in the conventional method [1,9]. The matrices α and β are defined by

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad (2.2.3)$$

where σ is the Pauli matrix and $\mathbf{1}$ is the unit matrix. Following standard methods this equation is modified for the rest energy to

$$(E + mc^2)\psi' = (c\alpha \cdot \pi + \beta mc^2 + eV)\psi', \quad (2.2.4)$$

where the modified spinor can be expressed as two two-component spinors, ψ_A and ψ_B . The minimal prescription is expressed through [11] the pure real,

$$\pi = \mathbf{p} - \frac{e}{c}\mathbf{A}, \quad (2.2.5)$$

and the Dirac equation splits into two interlinked equations,

$$\psi_B = \frac{c\sigma \cdot \pi}{E + 2mc^2 - eV} \psi_A, \quad (2.2.6)$$

$$(E - eV) \psi_A = \frac{c^2(\sigma \cdot \pi)^2}{E + 2mc^2 - eV} \psi_A. \quad (2.2.7)$$

One of these is an equation in ψ_A and the other links ψ_A to ψ_B . The second can be expressed as the wave equation,

$$H\psi_A = E\psi_A, \quad (2.2.8)$$

where H is the Hamiltonian,

$$H = \frac{c^2(\sigma \cdot \pi)^2}{E + 2mc^2 - eV} + eV. \quad (2.2.9)$$

This standard textbook procedure evidently gives a satisfactorily Hermitian equation which gives real and physical energy eigenvalues, positive and negative [1], but pure real.

If we let A be complex, we can write two Dirac equations,

$$(E - eV) \psi_A = c\sigma \cdot \pi \psi_B, \quad (2.2.10)$$

$$(E + 2mc^2 - eV) \psi_B = c\sigma \cdot \pi \psi_A, \quad (2.2.11)$$

$$(E - eV) \psi_A = c\sigma \cdot \pi^* \psi_B, \quad (2.2.12)$$

$$(E + 2mc^2 - eV) \psi_B = c\sigma \cdot \pi^* \psi_A, \quad (2.2.13)$$

and we can evaluate the consequences of this working hypothesis. Using Eqs. (2.2.10) and (2.2.13); or using (2.2.11) and (2.2.12) we obtain in both cases,

$$H\psi_A = E\psi_A, \quad (2.2.14)$$

$$H = \frac{c^2(\sigma \cdot \pi)(\sigma \cdot \pi^*)}{E + 2mc^2 - eV} + eV. \quad (2.2.15)$$

This equation is identical to Eq (26) of Ref. 10 except for a change of sign in V and a factor 2 multiplying mc^2 in the denominator. Using the standard [1] non-relativistic approximation,

$$E - eV \ll 2mc^2, \quad (2.2.16)$$

Eq. (2.2.15) can be written as,

$$H = H_1 + H_2 + \dots, \quad (2.2.17)$$

$$H_1 := \frac{e^2}{2mc^2} A \cdot A^*, \quad (2.2.18)$$

$$H_2 := \frac{e^2}{2mc^2} i\sigma \cdot A \times A^*. \quad (2.2.19)$$

The term labeled H_1 was first derived [10] by Volkov in 1935 and is pure real. It is the second order contribution of the electromagnetic field to the kinetic energy of the particle in the non-relativistic limit. The dot product $\mathbf{A} \cdot \mathbf{A}^*$ is often referred to in classical electrodynamics as a time average over many cycles. Therefore the working hypothesis that \mathbf{A} can be complex in Eqs. (2.2.14) and (2.2.15) leads to a standard Volkov result [10] for the theory of the Dirac fermion in the classical electromagnetic field.

The same hypothesis also leads to a novel term,

$$H_2 := \frac{e^2}{2mc^2} i\boldsymbol{\sigma} \cdot \mathbf{A} \times \mathbf{A}^*, \quad (2.2.20)$$

which in *S.I.* units [10] becomes

$$H_2 = \frac{e^2}{2m} i\boldsymbol{\sigma} \cdot \mathbf{A} \times \mathbf{A}^*, \quad (2.2.21)$$

and is also *pure real*. Thus, Volkov's term H_1 is accompanied by the term H_2 , which was first proposed using different methods in Ref 10.

2.3 The H_2 Term And Its Physical Meaning

The H_2 term represents a coupling between the half integral spin $\hbar\boldsymbol{\sigma}/2$ of the fermion and the conjugate product of the classical field, $\mathbf{A} \times \mathbf{A}^*$. It can be shown straightforwardly [10] that this is proportional to I/ω^2 ,

$$H_2 = \frac{e^2 c^2 B^{(0)2}}{2m\omega^2} i\boldsymbol{\sigma} \cdot \mathbf{k}. \quad (2.2.22)$$

Therefore it allows the possibility of resonance between states of the spinor induced not by a magnet but by the conjugate product $\mathbf{A} \times \mathbf{A}^*$ [3—6]. The resonance frequency is given in *S.I.* units by,

$$f_{\text{res}} = \left(\frac{e^2 \mu_0 c}{2\pi \hbar m} \right) \frac{I}{\omega^2}, \quad (2.2.23)$$

where μ_0 is the vacuum permeability, and for a given *S.I.* increases as the inverse square of ω . As shown in Ref. 10 this property is potentially of great usefulness if developed experimentally. The theory also reproduces the order of magnitude of optical *NMR* shifts [12] introduced at visible frequencies.

2.4 Discussion

The Volkov term \square_1 cannot be produced from the Dirac equation if we use the standard approach, that \mathbf{A} is real. Yet it is a term which has made its way into a standard textbook such as that of Itzykson and Zuber [13]. The standard theory produces a term proportional to $\mathbf{A} \cdot \mathbf{A}$, which is highly oscillatory for electromagnetic radiation, and which is zero at high frequencies. Yet it is well known [14] that there exist non-linear optical effects proportional to the square of potential and field quantities. One of these is the inverse Faraday effect, which is static magnetization by a circularly polarized electromagnetic field, and which is described phenomenologically with $\mathbf{A} \times \mathbf{A}^*$ [15]. There are therefore internal inconsistencies in the standard fermion-field theory of the Dirac equation.

In the standard approach, in which \mathbf{A} is pure real, the resonance frequency described by Eq. (2.2.23) becomes proportional to I/ω ,

$$f_{\text{res}} = \left(\frac{\mu_0}{4\pi c} \frac{e^2}{m^2} \right) \frac{I}{\omega}, \quad (2.2.24)$$

and so it is easily possible in theory to test the working hypothesis on which is based Eqs. (2.2.10) to (2.2.13). If one is not allowed to use a complex A then the resonance frequency is proportional to I/ω ; otherwise it is proportional to I/ω^2 . A simple beam experiment ought to be able to distinguish between these predictions experimentally, or to show that both are correct. In any event, the experimental demonstration of radiation induced fermion resonance would be of great practical value, and the theory of this effect has been developed elsewhere [16].

Acknowledgments

The Alpha Foundation is thanked for the award of a praesidium membership, honoris causa, and many colleagues for interesting discussions.

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