Paper 3

Infinitesimal Field Generators

The concept of infinitesimal field generator is introduced, using the principle that the underlying symmetry of special relativity is described by the Poincaré group, or ten generator inhomogeneous Lorentz group. This concept leads straightforwardly to the vacuum Maxwell equations and six cyclical relations between field components. Both the Maxwell equations and the field relations are given by the *E(2)* little group. The recently inferred *field spin*, the vacuum magnetic flux density labeled $\boldsymbol{B}^{(3)}$ in the complex basis ((1), (2), (3)), is rigorously non-zero from first principles. The existence of $\boldsymbol{B}^{(3)}$ is compatible with the vacuum Maxwell equations, and the cyclic relations are automatically covariant. The B cyclics are invariants of the classical field.

Key words: Infinitesimal field generators; $\boldsymbol{B}^{(3)}$ field; invariance of B cyclics.

3.1 Introduction

The concept is introduced at the classical level of infinitesimal generators of the vacuum electromagnetic field, and it is shown that the Maxwell equations can be deduced from this concept along with cyclical field relations involving the recently inferred *field spin* labeled $\boldsymbol{B}^{(3)}$ in the complex basis ((1), (2), (3)) [1—20]. It follows that the $\boldsymbol{B}^{(3)}$ field is rigorously non-zero if we accept our first principle, which asserts that the infinitesimal field generators are generators of the inhomogeneous Lorentz group. The antisymmetric matrix of field generators is interpreted in terms of intrinsic spin. The theory is automatically Lorentz covariant (independent of any frame of reference) and therefore compatible with the principle of relativity, and puts the theory of the classical electromagnetic field into close correspondence with Wigner's theory [21] of particles, a theory which ascribes to each particle an intrinsic mass and spin.

Section 3.2 describes the relativistic theory of spin angular momentum, and using this framework proposes an equivalence between the infinitesimal generators used to define this theory and novel infinitesimal generators of magnetic flux density and electric field strength in vacuo. The infinitesimal translation generator is made equivalent to the infinitesimal generator of a fully covariant vector potential. The Pauli-Lubanski operator of the relativistic spin angular momentum theory [22] is equivalent to the vector introduced by Afanasiev *et al.* [23] which is formed by multiplying the matrix of infinitesimal field operators with the infinitesimal generator of potential.

Section 3.3 is a straightforward deduction of the Lie algebra of field generators in the light-like condition. This algebra has the *E(2)* symmetry — that of a well known little group of the Poincaré group [22]. The *E(2)* symmetry is that of commutator relations between the novel infinitesimal field generators. Particular solutions of this Lie algebra are shown straightforwardly to be consistent with the vacuum Maxwell equations, which are thereby *deduced* from our first principle — that the underlying symmetry group of special relativity is the Poincaré group. The same *E(2)* Lie algebra gives six cyclical relations between eigenvalues of the field generators in the basis ((1), (2), (3)) [1—10]. Of these, three form the B

Cyclic theorem [1—10] in the light-like condition. It is easily shown that the B Cyclic theorem retains its form in the rest-frame (the hypothetical rest frame corresponding to a massive photon), and so the theorem is a Lorentz *invariant*. The algebra shows conclusively that $\boldsymbol{B}^{(3)}$ is rigorously non-zero from the first principle adopted for this development, i.e., that the Poincaré group is the symmetry group of special relativity. If one accepts this first principle, it follows that $\boldsymbol{B}^{(3)}$, the *field spin*, is rigorously non-zero. It is difficult to see how the Poincaré group cannot be the symmetry group of special relativity, and it is concluded that $\boldsymbol{B}^{(3)}$ is the fundamental spin of the classical electromagnetic field in vacuo. It has been shown recently [1—20] that $\boldsymbol{B}^{(3)}$ is only one of several possible longitudinal solutions in vacuo of the Maxwell equations, which as this work shows, are possible conservation equations compatible with the *E(2)* Lie algebra of infinitesimal field generators in the light-like condition. The antisymmetric matrix of generators is an intrinsic spin of the classical electromagnetic field, corresponding to the intrinsic spin angular momentum of the photon. The latter is described by the well known antisymmetric matrix $J_{\mu\nu}$ of relativistic angular momentum theory [22], and by the Pauli-Lubanski pseudo four- vector W^{μ} [22].

Finally we discuss this result in terms of a precise correspondence between intrinsic field spin and intrinsic photon spin in quantum field theory.

3.2 Relativistic Spin Angular Momentum Theory and Infinitesimal Field Generators

The theory of relativistic spin angular momentum for particles is developed for example by Barut [24] and Ryder [22] and relies on the Pauli-Lubanski pseudo four-vector. The latter is dual in four dimensions to the antisymmetric spin angular momentum tensor, and cannot be defined without the introduction of the energy-momentum four-vector. The Pauli-Lubanski four- vector is therefore,

$$W^{\lambda} := -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}p_{\mu}J_{\nu\rho}\,, \tag{2.3.1}$$

where $\epsilon^{\lambda\mu\nu\rho}$ (with $\epsilon^{0123} = 1$) is the antisymmetric unit four-tensor. The antisymmetric natrix $J_{\nu\rho}$ is given by,

$$J_{\nu\rho} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{bmatrix}, \qquad (2.3.2)$$

where every element is an element of spin angular momentum in four dimensions. The energy-momentum four-vector is defined as usual by,

$$p^{\mu} = \left(p^0, \boldsymbol{p}\right) = \left(\frac{En}{c}, \boldsymbol{p}\right). \qquad (2.3.3)$$

The Pauli-Lubanski pseudo four-vector is therefore a four dimensional cross product of angular and linear momentum for a classical particle.

It is well known [23,24] that these considerations can be extended to operators, infinitesimal generators of the Poincaré group (ten parameter inhomogeneous Lorentz group). In the operator representation $J_{\nu\rho}$ becomes a matrix of infinitesimal generators, of rotation generators J and boost generators K. The p_μ vector becomes the infinitesimal generator of translation in four dimensions [22,24]. The infinitesimal generators can be represented as matrices or as differential operator combinations [22]. The Pauli-Lubanski operator (W^{μ}) then becomes a product of the $J_{\nu\rho}$ and p_μ operators. Barut [24] shows that the Lie algebra of the W^{μ} operators is,

$$\left[W^{\mu}, W^{\nu}\right] = -i\epsilon^{\mu\nu\sigma\rho}p_{\sigma}W_{\rho}, \qquad (2.3.4)$$

which is a four dimensional commutator relation. The theory is automatically covariant and therefore compatible with the principle of special relativity, that the laws of physics are frame independent. Equation

(2.3.4) gives the Lie algebra of intrinsic spin angular momentum, because rotation generators are angular momentum operators within a factor $\hbar$ [22]. Similarly, translation generators are energy momentum operators within a factor $\hbar$. This development leads to a straightforward particle interpretation after quantization and to Wigner's famous result that every particle has intrinsic spin, including the photon.

Our basic ansatz is to assume that this theory applies to the vacuum electromagnetic field, considered as a physical entity of space-time in the theory of special relativity. The intrinsic spin of the classical electromagnetic field is the magnetic flux density $\boldsymbol{B}^{(3)}$ [1—20]. Infinitesimal generators of rotation correspond with those of intrinsic magnetic flux density in vacuo; those of boost with intrinsic electric field strength; those of translation with intrinsic, fully covariant, field potential. Thus, the symbols are transmuted as follows,

$$J \rightarrow B, \qquad K \rightarrow E, \qquad P \rightarrow A. \qquad (2.3.5)$$

In Cartesian notation, the Pauli-Lubanski vector of the particle theory becomes a pseudo four-vector operator of the classical electromagnetic field in the vacuum,

$$W^{\lambda} = -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}A_{\mu}F_{\nu\rho}. \qquad (2.3.6)$$

The Lie algebra (2.3.4) becomes a Lie algebra of the field.

3.3 The *E(2)* Lie Algebra of The Field

If it is assumed that the electromagnetic field propagates at c in vacuo, then we must consider the Lie algebra (2.3.4) in a light-like condition. The latter is satisfied by a choice of (Appendix 3A),

$$A^{\mu} := \left(A^0, A_Z\right), \qquad A^0 = A_Z, \qquad (2.3.7)$$

corresponding in the particle interpretation to the light like translation generator,

$$p^{\mu} := \left(p^{0}, p_{Z}\right), \qquad p^{0} = p_{Z} . \tag{2.3.8}$$

The Pauli-Lubanski pseudo-vector of the field in this condition is,

$$\begin{aligned} W^{\mu} &= \left(A_Z B_Z, A_Z E_Y + A_0 B_X, - A_Z E_X + A_0 B_Y, A_0 B_Z\right) \\ &= A_0\left(B_Z, E_Y + B_X, -E_X + B_Y, B_Z\right), \end{aligned} \tag{2.3.9}$$

and the Lie algebra (2.3.4) becomes

$$\left.\begin{aligned} \left[B_X + E_Y, B_Y - E_X\right] &= i\,(B_Z - B_Z), \\ \left[B_Y - E_X, B_Z\right] &= i\,(B_X + E_Y), \\ \left[B_Z, B_X + E_Y\right] &= i\,(B_Y - E_X), \end{aligned}\right] \tag{2.3.10}$$

which has *E(2)* symmetry. In the particle interpretation Eqs. (2.3.9) and (2.3.10) correspond to,

$$W^{\mu} = \left(p_Z J_Z, p_Z K_Y + p_0 J_X, - p_Z K_X + p_0 J_Y, p_0 J_Z\right) \tag{2.3.11}$$

and

$$\left.\begin{aligned} \left[J_X + K_Y, J_Y - K_X\right] &= i\,(J_Z - J_Z) \\ \left[J_Y - K_X, J_Z\right] &= i\,(J_X + K_Y) \\ \left[J_Z, J_X - K_Y\right] &= i\,(J_Y - K_X) \end{aligned}\right\} . \tag{2.3.12}$$

In the hypothetical rest frame the field and particle Pauli-Lubanski vectors are respectively

$$W^{\mu} = \left(0, A_0 B_X, A_0 B_Y, A_0 B_Z\right), \tag{2.3.13}$$

and

$$W^{\mu} = \left(0, P_0 J_X, P_0 J_Y, P_0 J_Z\right), \tag{2.3.14}$$

and the rest frame Lie algebra for field and particle is respectively (normalized $B^{(0)} = 1$ units)

$$\left[B_X, B_Y\right] = i B_Z \qquad (et\ cyclicum)\ , \tag{2.3.15}$$

and

$$\left[J_X, J_Y\right] = i J_Z, \qquad (et\ cyclicum)\ . \tag{2.3.16}$$

It is straightforward to show that the *E(2)* field Lie algebra (2.3.10) is compatible with the vacuum Maxwell equations written for eigenvalues of our novel infinitesimal field operators. This is demonstrated as follows.

A particular solution of the *E(2)*, or little group, Lie algebra (2.3.10) is given by equating infinitesimal field generators as follows,

$$B_Y = E_X, \qquad B_X = -E_Y\ . \tag{2.3.17}$$

It is assumed that the eigenfunction (ψ) operated upon by these infinitesimal field generators is such that the same relation (2.3.17) holds between eigenvalues of the field. In order for this to be true the eigenfunction must be the de Broglie eigenfunction, i.e., the phase of the classical electromagnetic field,

$$\psi = e^{i(\omega t - \kappa Z)}, \tag{2.3.18}$$

where ω is the frequency at instant t and κ the wavevector at point Z. This is demonstrated in Appendix 3B.

The relation (2.3.17) interpreted as one between eigenvalues is compatible with the plane wave solution of Maxwell's vacuum equations [1—10] for circular polarization, i.e.,

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}} (\boldsymbol{i} - i\boldsymbol{j}) e^{i(\omega t - \kappa Z)}, \tag{2.3.19}$$

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}} (i\boldsymbol{i} + \boldsymbol{j}) e^{i(\omega t - \kappa Z)}, \tag{2.3.20}$$

and this conveniently introduces the complex basis ((1), (2), (3)) defined by the unit vectors [1—10],

$$\left.\begin{aligned} \boldsymbol{e}^{(1)} &= \boldsymbol{e}^{(2)*} := \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}) \\ \boldsymbol{e}^{(3)} &= \boldsymbol{e}^{(3)*} := \boldsymbol{k} \end{aligned}\right\} . \tag{2.3.20}$$

It is concluded that our basic ansatz is compatible with Maxwell's vacuum equations, which are one possible way of ensuring that Eq. (2.3.17) holds.

It follows that the same analysis can be applied to the particle interpretation, giving,

$$\partial_\mu J^{\mu\nu} = \partial_\mu \tilde{J}^{\mu\nu} = 0, \tag{2.3.21}$$

in the vacuum. This is a possible conservation equation (simple relation between spins) which is compatible with the *E(2)* symmetry of the little group. In the particle interpretation this little group symmetry is the one

given by considering the most general Lorentz transform that leaves the light-like vector (2.3.8) invariant.

It is concluded that the vacuum Maxwell equations for the field correspond with Eq. (2.3.21) for the particle, an equation which asserts that the spin angular momentum matrix is divergentless. In vector notation we obtain from Eqs. (2.3.17) to (2.3.21) the familiar *S.I.* equations,

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{0}, \\ \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} &= 0, \\ \nabla \cdot \boldsymbol{E} &= 0, \end{aligned}\right\} \tag{2.3.22}$$

and the less familiar relation between eigenvalues of spin angular momentum in four dimensions,

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{J} &= 0, \\ \nabla \times \boldsymbol{J} + \frac{\partial K}{\partial t} &= \boldsymbol{0}, \\ \nabla \times \boldsymbol{K} - \frac{\partial \boldsymbol{J}}{\partial t} &= 0, \\ \nabla \cdot \boldsymbol{K} &= 0. \end{aligned}\right\} \tag{2.3.23}$$

Another particular solution of the *E(2)* Lie algebra (2.3.10) is given by commutator relations, of which there are six in total. Three of these form the recently inferred B cyclic theorem [1—10] ($B^{(0)} = 1$ units),

$$\left.\begin{aligned}\left[B_X, B_Y\right] &= iB_Z,\\ \left[B_Y, B_Z\right] &= iB_X,\\ \left[B_Z, B_X\right] &= iB_Y,\end{aligned}\right\} \tag{2.3.24}$$

and the other three are

$$\left.\begin{aligned}\left[E_X, E_Y\right] &= -iB_Z,\\ \left[B_Z, E_X\right] &= iE_Y,\\ \left[E_Y, B_Z\right] &= iE_X.\end{aligned}\right\} \tag{2.3.25}$$

In the particle interpretation these are parts of the Lie algebra [22,24] of rotation and boost generators of the Poincaré group,

$$\left.\begin{aligned}\left[J_X, J_Y\right] &= iJ_Z,\\ \left[J_Y, J_Z\right] &= iJ_X,\\ \left[J_Z, J_X\right] &= iJ_Y,\end{aligned}\right\} \tag{2.3.26}$$

$$\left.\begin{aligned}\left[K_X, K_Y\right] &= -iJ_Z,\\ \left[J_Z, K_X\right] &= iK_Y,\\ \left[K_Y, J_Z\right] &= iK_X.\end{aligned}\right\} \tag{2.3.27}$$

Using the methods sketched in Appendix 3B, we obtain from the Lie algebra of generators the following *S.I.* unit cyclic relations between field eigenvalues,

$$\left.\begin{aligned}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} &= iB^{(0)}\boldsymbol{B}^{(3)*},\\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*},\\ \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*},\end{aligned}\right\} \tag{2.3.28}$$

$$\left.\begin{aligned}\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} &= ic^2B^{(0)}\boldsymbol{B}^{(3)*},\\ \boldsymbol{B}^{(3)} \times \boldsymbol{E}^{(1)} &= icB^{(0)}\boldsymbol{E}^{(2)*},\\ \boldsymbol{B}^{(3)} \times \boldsymbol{E}^{(2)} &= -icB^{(0)}\boldsymbol{E}^{(1)*},\end{aligned}\right\} \tag{2.3.29}$$

where $\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)}$. Similarly in the particle interpretation, and switching from rotation generator to spin angular momentum we obtain,

$$\left.\begin{aligned}\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} &= i\hbar\boldsymbol{J}^{(3)*},\\ \boldsymbol{J}^{(2)} \times \boldsymbol{J}^{(3)} &= i\hbar\boldsymbol{J}^{(1)*},\\ \boldsymbol{J}^{(3)} \times \boldsymbol{J}^{(1)} &= i\hbar\boldsymbol{J}^{(2)*}.\end{aligned}\right\} \tag{2.3.30}$$

In the latter set of relations, $\hbar$ is the quantum of spin angular momentum.

In the hypothetical rest frame, we obtain for field and particle respectively, Eqs. (2.3.28) and (2.3.30), i.e., there are no boost generators as expected for a rest frame. The latter is hypothetical because an object translating at c identically does not have a rest frame by definition. We must therefore imagine an object translating infinitesimally close to c in vacuo in order to be able to back transform into a rest frame. This object can be thought of in our development as the electromagnetic field concomitant to photon with mass. In our new analysis the field and photon become topologically the same thing.

It is concluded that the $\boldsymbol{B}^{(3)}$ component is identically non-zero, otherwise all the field components vanish in the Lie algebra (2.3.24). If we assume Eq. (2.3.17), and at the same time assume that $\boldsymbol{B}^{(3)}$ is zero, then the Pauli-Lubanski pseudo four-vector (2.3.9) vanishes for all A_0. Similarly in the particle interpretation if we assume the equivalent of Eq. (2.3.17) and assume that $\boldsymbol{J}^{(3)}$ is zero, the Pauli-Lubanski vector W^{μ} vanishes. This is contrary to the definition of the helicity of the photon [22]. Therefore for finite field helicity we need a finite $\boldsymbol{B}^{(3)}$. The latter result is also indicated experimentally in magneto-optics [1—10], which can be used to observe the product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$. If $\boldsymbol{B}^{(3)}$ were zero this product would be zero, contrary to experience.

Therefore finite electromagnetic field helicity requires a finite $\boldsymbol{B}^{(3)}$ field in the light like condition. In the hypothetical rest frame a zero $\boldsymbol{B}^{(3)}$ would mean a zero $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$.

3.4 Discussion

The precise correspondence between field and photon interpretation of vacuum electromagnetism developed here indicates that *E(2)* symmetry does not imply that $\boldsymbol{B}^{(3)}$ is zero, any more than it implies $\boldsymbol{J}^{(3)} = \boldsymbol{0}$. The assertion $\boldsymbol{B}^{(3)} = \boldsymbol{0}$ is counter indicated by magneto-optical data, and the B cyclics (2.3.28) are Lorentz covariant, being part of a Lorentz covariant Lie algebra. Furthermore, if one assumes the particular solution (2.3.24) and (2.3.25), and uses in it the particular solution (2.3.17), we obtain from the three cyclics Eq. (2.3.25) the cyclics (2.3.24), i.e., we obtain,

$$\left.\begin{aligned} \left[B_Y, -B_X\right] &= iB_Z, \\ \left[B_Z, B_Y\right] &= -iB_X, \\ \left[B_Z, -B_X\right] &= -iB_Y. \end{aligned}\right\} \tag{2.3.31}$$

This is also the relation obtained in the hypothetical rest frame. Therefore the B cyclic theorem is a Lorentz *invariant* in the sense that it is the same in the rest frame and in the light-like condition.

This result can be checked by applying the Lorentz transformation rules for magnetic fields term by term, i.e., to $B^{(0)}$, $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ by considering a Lorentz boost at c in Z. The term $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is

transformed into itself multiplied by an indeterminate (0/0) which from the Lie algebra considered above is unity. The term $\boldsymbol{B}^{(3)}$ is unchanged, and $B^{(0)}$ must therefore be unchanged if we take the indeterminate to be unity. In the quantum interpretation,

$$B^{(0)} = \frac{\hbar\omega^2}{ec^2} = \frac{(\hbar\omega)\,\omega}{ec^2} = \left(\frac{\omega^2}{ec^2}\right)\hbar, \tag{2.3.32}$$

where e is the quantum of charge and c the speed of light in vacuo. If we consider $\hbar\omega$ to transform as energy and ω to transform as frequency [6] then $B^{(0)}$ is invariant under a Lorentz boost in Z. Since $\hbar\omega$ is the quantum of energy then it transforms as energy. Therefore it is concluded that the B cyclic theorem is invariant under a boost at c in Z. It appears unchanged in the Lie algebra of the light-like condition and of the rest frame as discussed already. (For intermediary boosts, taking place at v from the hypothetical rest frame, the numerical value of $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is unchanged, but it becomes a function of v. The product $iB^{(0)}\boldsymbol{B}^{(3)*}$ is invariant again.)

It is concluded that the B cyclic theorem in the field interpretation is a Lorentz invariant construct. The equivalent of this result in the particle interpretation for spin angular momentum is that the J cyclic theorem,

$$\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} = i\hbar\boldsymbol{J}^{(3)*}, \tag{2.3.33}$$

is a Lorentz invariant. This is compatible with the fact that $\hbar$ is an invariant and that $\boldsymbol{J}^{(3)}$ is invariant to a boost in Z. Thus $\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)}$ is invariant.

It is concluded overall that the ansatz adopted in this work is compatible both with the vacuum Maxwell equations and with the recently inferred cyclic relations between field components in vacuo [1—20]. As a result of this ansatz the $\boldsymbol{B}^{(3)}$ component in the field interpretation is non-zero in the light-like condition and in the rest frame, and is a solution of Maxwell's equations in the vacuum. The B cyclic theorem is a Lorentz invariant, and the product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is an experimental observable [1—10]. In this representation, $\boldsymbol{B}^{(3)}$ is a phaseless and fundamental field spin, an intrinsic property of the field in the same way as $\boldsymbol{J}^{(3)}$ is an intrinsic property of the photon. The scalar $B^{(0)}$ for the field plays the role of $\hbar$ for the photon, and if $\hbar\omega$ transforms as energy and ω as frequency, is also a Lorentz invariant. It is incorrect to infer from the Lie algebra (2.3.10) that $\boldsymbol{B}^{(3)}$ must be zero for plane waves. For the latter we have the particular choice (2.3.17) and the algebra (2.3.10) reduces to,

$$i(B_Z - B_Z) = 0, \tag{2.3.34}$$

which does not indicate that B_Z is zero any more than the equivalent particle interpretation indicates that J_Z is zero. That B_Z is zero is therefore a wholly unwarranted assumption of the literature [22]. Vacuum electromagnetism is *not* purely transverse in nature, and this result has recently been shown in several different ways [1—20]. By using the Poincaré group for vacuum electromagnetism it becomes easier to unify field theory [1—20], this particular paper has introduced the notion of infinitesimal field generators and has shown that this ansatz is compatible both with the vacuum Maxwell equations and the B cyclic theorem. The latter is a fundamental theorem of fields which shows that transverse solutions are always accompanied by longitudinal solutions.

Acknowledgments

It is a pleasure to acknowledge numerous Internet discussions and to thank the Alpha Foundation of the Institute of Physics, Budapest, Hungary, for an honorary membership.

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Appendix 3A. Basics of Poincaré Group Electrodynamics

The basic ansatz used in the text is that there exists a field vector analogous to the Pauli-Lubanski vector of particle physics; a field vector defined by,

$$W^{\lambda} := \tilde{F}^{\lambda\mu} A_{\mu}, \tag{2.3A.1}$$

where $\tilde{F}^{\lambda\mu}$ is the dual of the antisymmetric field tensor. Without assumptions of any kind, this vector has components,

$$\begin{aligned} W^0 &= -B^1A_1 - B^2A_2 - B^3A_3, \\ W^1 &= B^1A_0 + E^3A_2 - E^2A_3, \\ W^2 &= B^2A_0 - E^3A_1 + E^1A_3, \\ W^3 &= B^3A_0 + E^2A_1 - E^1A_2. \end{aligned} \tag{2.3A.2}$$

If we assume: a) that for the transverse components,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{2.3A.3}$$

if, b) B and A are plane waves,

$$\begin{aligned} \boldsymbol{A} &= \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \\ \boldsymbol{B} &= \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \end{aligned} \tag{2.3A.4}$$

and: c) if the longitudinal $\boldsymbol{E}^{(3)}$ is zero, then Eqs. (2.3A.2) reduce to those used in the text, i.e.,

$$\begin{aligned} W_0 &= A_Z B_Z, \\ W_X &= A_0 B_X + A_Z E_Y, \\ W_Y &= A_0 B_Y - A_Z E_X, \\ W_Z &= A_0 B_Z. \end{aligned} \tag{2.3A.5}$$

These assumptions mean that

$$A^{\mu} := \left(A^0, 0, 0, A^3\right), \qquad A^0 = A^3, \tag{2.3A.6}$$

can be used as an ansatz. Conversely, use of this definition means that the transverse components are plane waves and for the transverse components $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

In the Coulomb gauge the vector W^{μ} vanishes, meaning that there is no correspondence between particle and field theory for the Coulomb gauge or traditional assumption of transversality. Our final result is,

$$W^{\mu} = A^0\left(B_Z, 0, 0, B_Z\right), \tag{2.3A.7}$$

which is compatible with the *E(2)* Lie algebra of the text and with the vacuum Maxwell equations; together with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ for transverse components and,

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{2.3A.8}$$

for longitudinal ones. It is significant that the $K^{(3)}\left(= K_Z\right)$ generator does not appear in the Lie algebra *E(2)*. This does not mean that $\boldsymbol{E}^{(3)}$ is zero necessarily, but it does not play the same role as $\boldsymbol{B}^{(3)}$. The latter is the most fundamental field spin, i.e., intrinsic spin of the classical electromagnetic field.

Appendix 3B. Inference of The De Broglie Wavefunction From Eq (2.3.17)

Equation.(2.3.17) of the text equates differential operators, i.e., field generators. The B operator is directly proportional to the J operator, the E operator to the K operator. In the vacuum $E^{(0)} = cB^{(0)}$ in *S.I.* units. These operate on a function such that the eigenvalues are related in the same way as the operators. Define the B and E differential operators by,

$$B_X\psi := -iB^{(0)}\left(Y\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial Y}\right)\psi ,$$
$$E_Y\psi := -iE^{(0)}\left(t\frac{\partial}{\partial X} + X\frac{\partial}{\partial t}\right)\psi , \tag{2.3B.1}$$

and it is clear that the wavefunction is

$$\psi = e^{i(\omega t - \kappa Z)} , \tag{2.3B.2}$$

where $\kappa = \omega/c$. This is the well known phase of the vacuum electromagnetic wave, known sometimes as the de Broglie wavefunction.

Appendix 3C. Commutators to Cyclics

In order to translate a Cartesian commutator relation such as

$$\left[B_X, B_Y\right] = iB^{(0)}B_Z , \tag{2.3C.1}$$

to a ((1), (2), (3)) basis vector equation such as,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} , \tag{2.3C.2}$$

consider firstly the usual unit vector relation in the Cartesian frame,

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} . \tag{2.3C.3}$$

The unit vector $\boldsymbol{i}$ for example is defined by

$$\boldsymbol{i} := u_x \boldsymbol{i} , \tag{2.3C.4}$$

where u_x is a rotation generator [22] in general a matrix component. Therefore,

$$u_X = i\,(J_X)_{YZ} . \tag{2.3C.5}$$

The cross product $\times \boldsymbol{j}$ therefore becomes a commutator of matrices,

$$\left[J_X, J_Y\right] = iJ_Z , \tag{2.3C.6}$$

i.e.,

$$\left[J_X, J_Y\right] :=$$

$$\frac{1}{i}\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\frac{1}{i}\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}-\frac{1}{i}\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}\frac{1}{i}\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix} \tag{2.3C.7}$$

$$= \begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix} := iJ_Z .$$

This can be extended immediately to angular momentum operators and infinitesimal magnetic field generators. Thus, a commutator such as (2.3C.1) is equivalent to a vector cross product. If we write $B^{(0)}$ as the scalar magnitude of magnetic flux density, the commutator (2.3C.1) becomes the vector cross product,

$$(B^{(0)}\boldsymbol{i}) \times (B^{(0)})\boldsymbol{j} = B^{(0)}(B^{(0)}\boldsymbol{k}) , \tag{2.3C.8}$$

which can be written conveniently as,

$$(B_X B_Y)^{1/2}\boldsymbol{i} \times (B_X B_Y)^{1/2}\boldsymbol{j} = iB^{(0)}B_Z\boldsymbol{k} . \tag{2.3C.9}$$

However, the Cartesian basis can be extended to the circular basis using relations between unit vectors [1—10], so Eq. (2.3C.9) can be written in the circular basis as,

$$(B_X B_Y)^{1/2}\boldsymbol{e}^{(1)} \times (B_X B_Y)^{1/2}\boldsymbol{e}^{(2)} = iB^{(0)}B_Z\boldsymbol{e}^{(3)*} , \tag{2.3C.10}$$

which is equivalent to,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} , \tag{2.3C.11}$$

where we define

$$\boldsymbol{B}^{(1)} := (B_X B_Y)^{1/2}\boldsymbol{e}^{(1)} = \boldsymbol{B}^{(2)*} ,$$
$$\boldsymbol{B}^{(3)*} := B_Z\boldsymbol{e}^{(3)*} . \tag{2.3C.12}$$

To complete the derivation we multiply both sides of Eq. (2.3C.11) by the phase factor $e^{i\phi}e^{-i\phi}$ to obtain the B Cyclic theorem [1-10]. The latter is equivalent therefore to a commutator relation between infinitesimal magnetic field generators. Similarly,

$$\left[E_X, E_Y\right] = ic^2B^{(0)}B_Z , \tag{2.3C.13}$$

is equivalent to

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = ic^2B^{(0)}\boldsymbol{B}^{(3)*} , \tag{2.3C.14}$$

and so forth.