SOME NOTES ON DIFFERENTIAL GEOMETRY

Notation

$$I_1 = \oint_C F_x dx + F_y dy + F_z dz = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$I_2 = \oint_S G_x dy \wedge dz + G_y dz \wedge dx + G_z dx \wedge dy = \int \boldsymbol{G} \cdot d\boldsymbol{S}$$

The line and surface integrals are called chains, and the object integrated is called the differential form. Form are dual to chains.

C_0	$0-\text{chain} = \text{point}$
C_1	$1-\text{chain} = \text{line}$
C_2	$2-\text{chain} = \text{area}$
C_3	$3-\text{chain} = \text{volume}$
C_n	$n-\text{chain}$

The boundary of an n-chain is an n-1 chain. The boundary of an area is a line; the boundary of a line is two points.

The boundary operator = ∂, and maps C_n onto C_{n-1}.

$$\partial C_n = C_{n-1} \tag{1}$$

Some chains have no boundaries. The surface of a sphere is a 2-chain with no boundary. A closed line like a circle is a 1-chain with no boundary. Closed chains are called cycle, and denoted Z_n.

$$\partial Z_n = 0 \tag{2}$$

Z_n is the kernel of the mapping (1).

A closed surface B_2 is the boundary of a volume, and a closed line B_1 is the boundary of an area; so:

$$B_n = \partial C_{n+1} \tag{3}$$

$$\partial B_n = 0 \tag{4}$$

B_n is the image of the mapping (1).

The boundary of a boundary is zero:

$$\partial^2 = 0 \tag{5}$$

A chain which is a boundary is closed.

In Euclidean spaces, $Z_n = B_n$, but in general these are closed chains which are not boundaries, so:

$$Z_n \supset B_n \tag{6}$$

On the space of a circle, S', the circle itself is not the boundary of any part of the space, because the two dimensional area is not part of S', which is one dimensional.

The integral of a form over a chain is a number:

$$\int_{C_n} \omega_n = \int_{C_n} f_{i_1 \ldots\ldots i_n} dx_{i_1} \wedge dx_{i_2} \wedge \ldots . \wedge dx_{i_n} = \text{number} \tag{7}$$

A 1-form ω_1 is integrated over a line (1-chain), so in 3-D, space is of the form $Adx + Bdy + Cdz$.

$$\omega_2, \text{2-form}, f dx \wedge dy + g dy \wedge dz + h dz \wedge dx$$
$$\omega_3, \text{3-form}, F dx \wedge dy \wedge dz$$
$$dx \wedge dy = -dy \wedge dx; \qquad dx \wedge dx = 0 \qquad etc.$$

Exterior Derivative Operator

$$d\omega_n = \omega_{n+1} \tag{8}$$

$$\begin{aligned} d\left(Adx + Bdy + Cdz\right) &\equiv \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial A}{\partial z} dz \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy \\ &+ \frac{\partial B}{\partial z} dz \wedge dy + \frac{\partial C}{\partial x} dx \wedge dz + \frac{\partial C}{\partial y} dy \wedge dz \\ &= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy + \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx \end{aligned}$$

This is

$$\begin{gathered} \nabla \times \boldsymbol{F} \ \text{ if } \boldsymbol{F} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k} \\ d\boldsymbol{F} \equiv \nabla \times \boldsymbol{F} \end{gathered} \tag{9}$$

Similarly, if

$$\omega_2 = f dx \wedge dy + g dy \wedge dz + h dz \wedge dx$$
$$d\omega_2 = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}\right) dx \wedge dy \wedge dz$$

If $W \equiv (g, h, f)$;

$$\nabla \bullet W = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial f}{\partial z} \quad (10)$$

If ω is a p form and C is a p+1 chain, then:

$$\int_{\partial C} \omega = \int_{C} \partial\omega \quad (11)$$

For example, if $\omega_1 = F_x d_x + F_y d_y + F_z d_z$

$$\int_{\partial A} F_x dx + F_y dy + F_z dz$$

$$= \int_{A} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx \wedge dy + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) dz \wedge dx + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dy \wedge dz$$

i.e.

$$\oint_{\partial A} A \bullet dl = \int_{S} \nabla \times A \bullet dS \quad (12)$$

Application of Eqn. (11) to a Circle

A circle is a closed chain and is a cycle, so:

$$\int_{\partial Z_1} \omega = \int_{0} d\omega \quad (13)$$

The integration of ω over a circle results in a surface integration where the surface has shrunk to a point. Therefore:

$$\oint_{\partial Z_I} \omega = \int_{0} d\omega = 0 \quad (14)$$

For Integration over a Circle

$$\oint_{\partial A} A \bullet dl = \int_{S} \nabla \times A \bullet dS = 0 \quad (15)$$

This is the Stoke theorem used in U(1) electrodynamics.

The **Holonomy**:

$$\gamma = \exp\left(i \oint_{\partial A} A \bullet dl \right) \quad (16)$$

is the same in the Sagnac effect for A and C loops, and so $\Delta\phi = 0$. There is no phase difference for platform at rest for **any A** in U(1). The only possible explanation of the Sagnac effect is non-Abelian in nature.

The Non-Abelian Stokes Theorem

This is described by Broda in Barrett and Grimes p. 498 ff. It is expressed as a holonomy equation:

$$P\left(i\oint_{\partial S=C} A_i(x)dx^i\right) = P'\exp\left(\frac{i}{2}\int_S F'_{ij}(x)dx^i \wedge dx^j\right) \tag{17}$$

where P denotes path ordering, P' denotes surface ordering and F'_{ij} is a path dependent curvature:

$$F'_{ij}(x) = \mathrm{U}^{-1}(x,0); \qquad F_{ij}(x)\mathrm{U}(x,0) \tag{18}$$

where U(x,0) is the parallel transport operator along the path l in the surface joining the base point 0 of dS with the point x, i.e.:

$$U(x,0) = P\exp\left(i\int_l A_i(y)dy^i\right) \tag{19}$$

Here:

$$F_{ij} = \partial_i A_j - \partial_j A_i - i\left[A_i, A_j\right] \tag{20}$$

where

$$A_i = A_i^a T^a; \qquad T^{a+} = T^a; \qquad \left[T^a, T^b\right] = if^{abc}T^c$$

Application to Interferometry

We have:

$$\boldsymbol{A}_\mu = A_\mu^{(1)}e^{(1)} + A_\mu^{(2)}e^{(2)} + A_\mu^{(3)}e^{(3)} \tag{21}$$

$$\boldsymbol{G}_{\mu\nu} = G_{\mu\nu}^{(1)}e^{(1)} + G_{\mu\nu}^{(2)}e^{(2)} + G_{\mu\nu}^{(3)}e^{(3)} \tag{22}$$

We consider a line integral in the internal space:

$$\oint_{\partial S} A_\mu^{(1)}de^{(1)} + A_\mu^{(2)}de^{(2)} + A_\mu^{(3)}de^{(3)} = \oint_{\partial S} A_\mu^{(3)}de^{(3)} = \oint_{\partial S=C} A_3(x)dx^3$$

The integration takes place over a circle with a line perpendicular to the circle. Therefore:

$$\oint_{\partial S}\left(A_\mu^{(1)}de^{(1)} + A_\mu^{(2)}de^{(2)}\right) = \oint_{\partial Z_1}\left(A_\mu^{(1)}de^{(1)} + A_\mu^{(2)}de^{(2)}\right) = 0$$

The Sagnac effect is therefore described by:

$$P\exp\left(i\oint_{\partial S=C} A_3(x)dx^3\right) = P'\exp\left(\frac{i}{2}\int_S F'_{12}(x)dx^1\wedge dx^2\right)$$

The right hand side can be expressed as:

$$P'\exp\left(\frac{ig}{2}\int_S B^{(3)}dAr\right)$$

and the left hand side as:

$$P\exp\left(i\oint_{\partial S=C}\kappa_Z dZ\right)$$

General Result

Interferometry and Optics are described by:

$$P\exp\left(i\oint_{\partial S=C}\kappa_Z dZ\right) = P'\exp\left(\frac{ig}{2}\int_S B^{(3)}dAr\right)$$

and this is a major triumph for O(3) electrodynamics.