THE SAGNAC EFFECT

U(1) Holonomy Difference

It is shown as follows that the holonomy difference in the U(1) Yang-Mills theory of the Sagnac effect is zero. Consider the boundary:

$$x^2 + y^2 = 1 \tag{1}$$

of the assumed circular path of the light beam in the Sagnac effect. The line integral of a constant around this boundary vanishes:

$$\begin{aligned}\oint dr &= \int_0^{2\pi} dx + \int_0^{2\pi} dy \\ &= -\int_0^{2\pi} \sin\phi\, d\phi + \int_0^{2\pi} \cos\phi\, d\phi \\ &= 0 = -\oint dr\end{aligned} \tag{2}$$

Therefore:

$$\oint \kappa \cdot \boldsymbol{dr} = -\oint \kappa \cdot \boldsymbol{dr} = 0 \tag{3}$$

because $\boldsymbol{\kappa}$ is not a function of $\boldsymbol{r}$. Therefore:

$$\exp\left(i \oint_C \kappa \cdot \boldsymbol{dr} \right) = \exp\left(-i \oint_A \kappa \cdot \boldsymbol{dr} \right) = 1 \tag{4}$$

and the holonomy for A and C loops is equal. The holonomy difference and phase difference is **zero**. This is contrary to observation.

In the U(1) Yang-Mills gauge theory, the only vector potential is **transverse** to the path of the light beam:

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}} (\boldsymbol{ii} + \boldsymbol{j}) \exp^{i(\omega t - \kappa \cdot \boldsymbol{r})} \tag{5}$$

Therefore:

$$\boldsymbol{A}^{(1)} \perp \boldsymbol{r} \;\text{ and }\; \boldsymbol{A}^{(1)} \cdot \boldsymbol{r} = 0. \tag{6}$$

Therefore:

$$\oint \boldsymbol{A}^{(1)} \cdot \boldsymbol{dr} = 0 = -\oint \boldsymbol{A}^{(1)} \cdot \boldsymbol{dr} \tag{7}$$

and the holonomy for A and C (Wu-Yang phases) are the same:

$$\exp\left(i\oint_C A^{(1)}\bullet d\boldsymbol{r}\right)=\exp\left(-i\oint_A A^{(1)}\bullet d\boldsymbol{r}\right)=1 \tag{8}$$

Therefore there is no explanation of the Sagnac effect in Wu-Yang U(1) gauge field theory.

O(3) Holonomy Difference

In the O(3) Yang-Mills theory of the Sagnac Effect:

$$\boldsymbol{G}^{\mu\nu}=G^{\mu\nu(1)}\boldsymbol{e}^{(1)}+G^{\mu\nu(2)}\boldsymbol{e}^{(2)}+G^{\mu\nu(3)}\boldsymbol{e}^{(3)} \tag{9}$$

$$A^{\mu\nu}=A^{\mu(1)}\boldsymbol{e}^{(1)}+A^{\mu(2)}\boldsymbol{e}^{(2)}+A^{\mu(3)}\boldsymbol{e}^{(3)} \tag{10}$$

because the internal gauge space is regarded as the **physical** space of three dimensions, as represented by the basis ((1), (2), (3)).

Following Ryder p. 120, the effect of a round trip by parallel transport is given by the holonomy:

$$\begin{aligned}\gamma &=\exp\left(\iint\left[D_\mu,D_\nu\right]d\sigma^{\mu\nu}\right)\\ &=\exp\left(-ig\iint\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)d\sigma^{\mu\nu}\right)+\exp\left(-g^2\left[A_\mu,A_\nu\right]d\sigma^{\mu\nu}\right)\end{aligned}$$

If we choose the area to be defined by the X and Y directions of the Minkowski spacetime:

$$\begin{aligned}\gamma &=\exp\left(-ig\iint\left(\partial_X A_Y^{(3)}-\partial_Y A_X^{(3)}\right)dAr\right)+\exp\left(-g^2\iint\left|A^{(1)}\times A^{(2)}\right|dAr\right)\\ &=\exp\left(-i\kappa^2 Ar\right)=\exp\left(-ig\iint \boldsymbol{B}^{(3)}\bullet d\boldsymbol{Ar}\right)\end{aligned}$$

Under motion reversal symmetry:

$$T\left(\boldsymbol{B}^{(3)}\right)=-\boldsymbol{B}^{(3)}$$

so **the holonomy difference** is ϕ_2, which is observed as **the phase difference**:

$$\cos\left(\phi_2\pm 2\pi n\right)$$

with platform at rest.

The non-Abelian Stokes theorem shows that the holonomy is given by:

$$\exp\left(-i\oint\kappa_Z^{(2)}dZ\right)=\exp\left(-ig\iint \boldsymbol{B}^{(3)}\bullet d\boldsymbol{Ar}\right).$$

Unlike the U(1) case, this is the internal space ((1), (2), (3)) and the A-C holonomy difference is non-zero with platform at rest, as observed.

Effect of Rotating the Platform

In U(1) Yang-Mills theory, there is no explanation.

In U(1) Yang-Mills theory, we define:

$$\kappa^{\mu} = \kappa^{\mu(1)} e^{(1)} + \kappa^{\mu(2)} e^{(2)} + \kappa^{\mu(3)} e^{(3)}$$

In condensed notation, the effect of a phase transformation is

$$\kappa_{\mu} \rightarrow S \kappa_{\mu} S^{-1} - i\left(\partial_{\mu} S\right) S^{-1}$$

For a rotation about the Z axis $\perp$ plane of the platform:

$$S = \exp\left(i J_Z \alpha\left(x^{\mu}\right)\right)$$

$$\kappa_{\mu} \rightarrow \kappa_{\mu} \pm \partial_{\mu} \alpha$$

$$\omega \rightarrow \omega \pm \Omega; \qquad \Omega = \frac{\partial \alpha}{\partial t}.$$

This produces an extra phase difference:

$$\cos\left(4 \frac{\omega \Omega A r}{c^2} \pm 2\pi n\right)$$

as observed.