

## **DEVELOPMENT OF THE SACHS THEORY OF ELECTRODYNAMICS**

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The most general form of electrodynamics has been derived by Sachs [1] from the irreducible representations of the Einstein group. In this paper the Sachs theory is developed as a gauge theory with a vacuum four-current  $i_j^\mu$ . The B Cyclic Theorem O(3) electrodynamics is derived from a consideration of four-vectors appearing in the Sachs theory, and electromagnetic helicity, expressed in terms of the  $B^{(3)}$  field of O(3) electrodynamics, is derived from the more general Sachs theory.

Key words: Sachs's theory of electrodynamics, gauge theory, vacuum four-current, B cyclic theorem, helicity.

## 1. INTRODUCTION

The most general form of electrodynamics has been demonstrated by Sachs [1] to have a non-Abelian structure, which is developed in Sec. 2 into a gauge theory. Circumstances under which the Sachs theory reduces to O(3) electrodynamics are defined and the existence of  $\mathbf{B}^{(3)}$  field [2] is proven from the first principles of general relativity. In Sec. 3, the vacuum four-current that exists under all circumstances in the Sachs theory is developed and compared with vacuum currents inherent in O(3) electrodynamics and the Lehnert theory of electrodynamics [3]. In the Sachs theory, the helicity is not conserved because there is in-built parity violation [1].

## 2. ELECTROMAGNETIC GAUGE THEORY FROM GENERAL RELATIVITY

The field tensor in the Sachs theory has the form

$$F_{\mu\nu} = \partial_\mu A_\nu^* - \partial_\nu A_\mu^* + \frac{1}{8} \Phi R (q_\mu a_\nu^* - q_\nu a_\mu^*), \quad (1)$$

where

$$\begin{aligned} q_\mu &:= (q_\mu^{(0)}, q_\mu^{(1)}, q_\mu^{(2)}, q_\mu^{(3)}), \\ A_\mu^* &:= (-A_\mu^{(0)}, A_\mu^{(1)}, A_\mu^{(2)}, A_\mu^{(3)}), \end{aligned} \quad (2)$$

$q_\mu$  is a quaternion-valued metric,  $\Phi$  has the units of magnetic flux,  $R$  is the scalar curvature [1], and  $A_\mu^*$  is a quaternion-valued potential:

$$A_\gamma^* = \frac{\Phi}{4} q_\gamma^* \int (K_{\rho\lambda} q^\lambda + q^\lambda K_{\rho\lambda}^+) dx_\rho, \quad (3)$$

$K_{\rho\lambda}$  being the curvature tensor [1]. Here the upper indices denote the ((1), (2), (3)) basis used for the description of space in O(3) electrodynamics [4-8], and the index  $\gamma$  denotes the Cartesian basis (X, Y, Z).

The index (0) denotes the timelike part of the four-vector. It has been shown elsewhere that Eq. (1) can be written as a gauge field equation for electrodynamics:

$$F_{\mu\nu} = \partial_\mu A_\nu^* - \partial_\nu A_\mu^* - ig[A_\mu^*, A_\nu^*], \quad (4)$$

where  $g$  is a coefficient. Equation (4) shows that the most general form of electromagnetism is non-Abelian. This result is derived from a consideration of the irreducible representations of the Einstein group [1], and Eq. (4) is therefore the result of a holonomy in Riemannian spacetime. Therefore all the rules of gauge theory [9], which is a generalization of O(3) electrodynamics [4-8], apply. The latter has already been developed extensively in terms of gauge field theory [9], and in the following sections of this paper some of the derivations and results of O(3) electrodynamics are generalized with the Sachs theory, using Cartesian components of quaternion-valued four-vectors.

The most fundamental feature of O(3) electrodynamics is the existence of the  $\mathbf{B}^{(3)}$  field [4-8], which is longitudinally directed along the axis of propagation and which is defined in terms of the vector potential plane wave:

$$\mathbf{A}^{(1)} = \mathbf{A}^{(2)*}. \quad (5)$$

From the irreducible representations of the Einstein group, there exist four-vectors that are generally covariant and take the form:

$$\begin{aligned} B_1^\mu &= \left( B_X^{(0)}, B_X^{(1)}, B_X^{(2)}, B_X^{(3)} \right), \\ B_2^\mu &= \left( B_Y^{(0)}, B_Y^{(1)}, B_Y^{(2)}, B_Y^{(3)} \right), \\ B_3^\mu &= \left( B_Z^{(0)}, B_Z^{(1)}, B_Z^{(2)}, B_Z^{(3)} \right). \end{aligned} \quad (6)$$

All these components exist in general and the  $\mathbf{B}^{(3)}$  field can be identified as the  $B_Z^{(3)}$  component. In O(3) electrodynamics these four-vectors reduce to:

$$\begin{aligned} B_1^\mu &= \left( 0, B_X^{(1)}, B_X^{(2)}, 0 \right), \\ B_2^\mu &= \left( 0, B_Y^{(1)}, B_Y^{(2)}, 0 \right), \\ B_3^\mu &= \left( B_Z^{(0)}, 0, 0, B_Z^{(3)} \right), \end{aligned} \quad (7)$$

so it can be concluded that O(3) electrodynamics is developed in a curved spacetime which is defined in such a way that

$$\mathbf{B}^{(3)*} = -ig\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}. \quad (8)$$

In O(3) electrodynamics there exists the cyclic relation

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(3)} = iB^{(0)}\mathbf{B}^{(3)*}, \quad (9)$$

and in general relativity this cyclic relation can be developed with a cross product of four-vectors, using commutators such as

$$[B_i^\mu, B_j^\nu], \quad i = 1, 2, 3. \quad (10)$$

In this special case of O(3) electrodynamics, the vector

$$B_3^\mu = (B_Z^{(0)}, B_Z^{(1)}, B_Z^{(2)}, B_Z^{(3)}) \quad (11)$$

reduces to

$$B_3^\mu = (B_Z^{(0)}, 0, 0, B_Z^{(3)}), \quad (12)$$

which is obtainable from the duality relation

$$\tilde{B}^\mu = \frac{1}{2} E^{\mu\nu\sigma\rho} G_{\sigma\rho} E_\nu, \quad (13)$$

where  $E_\nu$  is a unit vector [10] and where  $G_{\sigma\rho}$  is the dual of  $\tilde{B}^\mu$ . Similarly, there exists in general the four-vector

$$A_3^\mu = (A_Z^{(0)}, A_Z^{(1)}, A_Z^{(2)}, A_Z^{(3)}), \quad (14)$$

which reduces in O(3) electrodynamics to

$$A_3^\mu = (A_Z^{(0)}, 0, 0, A_Z^{(3)}), \quad (15)$$

corresponding to linear momentum, but in the curved O(3) spacetime.

A more general definition of electromagnetic helicity can be given [11]:

$$a^0 = \int \mathbf{A} \cdot \mathbf{B} dV, \quad (16)$$

and in the Sachs theory this becomes

$$h := \int A_\mu^* \tilde{B}^\mu dV, \quad (17)$$

a quantity which is not conserved [1] because in the Sachs theory there is in-built parity violation. In the special case of O(3), the electrodynamics equation (17) reads

$$h(O(3)) = \int (-A_Z^{(0)}, 0, 0, A_Z^{(3)}) (B_Z^{(0)}, 0, 0, B_Z^{(3)}) dV \quad (18)$$

### 3. THE VACUUM FOUR-CURRENT

In the Sachs theory the vacuum four-current is quaternion-valued and has components such as

$$\begin{aligned} j_1^\mu &= \left( j_X^{(0)}, j_X^{(1)}, j_X^{(2)}, j_X^{(3)} \right), \\ j_2^\mu &= \left( j_Y^{(0)}, j_Y^{(1)}, j_Y^{(2)}, j_Y^{(3)} \right), \\ j_3^\mu &= \left( j_Z^{(0)}, j_Z^{(1)}, j_Z^{(2)}, j_Z^{(3)} \right), \end{aligned} \quad (19)$$

which in  $O(3)$  electrodynamics reduces to

$$\begin{aligned} j_1^\mu(O(3)) &= \left( 0, j_X^{(1)}, j_X^{(2)}, 0 \right), \\ j_2^\mu(O(3)) &= \left( 0, j_Y^{(1)}, j_Y^{(2)}, 0 \right), \\ j_3^\mu(O(3)) &= \left( j_Z^{(0)}, 0, 0, j_Z^{(3)} \right). \end{aligned} \quad (20)$$

The existence of a vacuum current such as this is indicated in  $O(3)$  electrodynamics by its inhomogeneous field equation

$$D_\mu \mathbf{G}^{\mu\nu} = \mathbf{J}^\nu, \quad (21)$$

which is a Yang-Mills type equation. The concept of vacuum current was introduced into Maxwell-Heaviside theory by Lehnert [12].

### 4. DISCUSSION

The concepts of  $O(3)$  electrodynamics [1-8] can be derived from general relativity by considering the types of four-vector used in this paper, where the upper indices indicate the basis ((1), (2), (3)) for space and the lower indices the (X, Y, Z) indices. The same structure of general relativity theory can be used [1] to show that in Maxwell-Heaviside theory the field does not propagate through curved spacetime, so  $O(3)$  electrodynamics is a particular case in which the theory is developed in conformally curved spacetime [13,14], allowing the field to propagate through the vacuum.

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