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Concerning Radii in Einstein's Gravitational Field

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Abstract It is demonstrated herein that the quantity 'r' appearing in the so-called "Schwarzschild solution" is neither a distance nor a geodesic radius but is in fact the radius of Gaussian curvature. The radius of Gaussian curvature does not generally determine the geodesic radial distance (the proper radius) from the centre of spherical symmetry of a spherically symmetric metric manifold except in the case of a Euclidean space.

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1 Introduction

In the usual interpretation of Hilbert's [1–3] "Schwarzschild's solution", the quantity r therein has *never* been properly identified. It has been variously called " the radius" [4,5] of a sphere, the "coordinate radius"[6] or "radial coordinate" [7,8] or "radial space coordinate" [9], the "areal radius" [6,10], the "reduced circumference" [11], and even a "*a gauge choice, which defines* r" [12]. In the particular case of $r=2GM/c^2$ it is invariably referred to as the "Schwarzschild radius" or the "gravitational radius". However, the irrefutable geometrical fact is that r, in Hilbert's version of the Schwarzschild/Droste line-element, is the radius of Gaussian curvature [13–16], and as such it *does not* in fact determine the geodesic radial distance from the centre of spherical symmetry located at an arbitrary point in the related metric manifold. Indeed, it does not in fact determine any distance at all in a spherically symmetric Riemannian metric manifold. It is the radius of Gaussian curvature merely by virtue of its formal geometric relationship to the Gaussian curvature.

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It must also be emphasized that a geometry is completely determined by the *form* of its line-element [17].

2 Gaussian curvature

Recall that Hilbert's version of the "Schwarzschild" solution is (using $c=G=1$),

$$ds^2=\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2 -r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{1}$$

wherein r can, by assumption (i.e. without proof), in some way or another, go down to zero. Schwarzschild's [18] actual solution, for comparision, is

$$ds^2=\left(1-\frac{\alpha}{R}\right)dt^2-\left(1-\frac{\alpha}{R}\right)^{-1}dR^2 -R^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{2}$$

$$R=R(r)=\left(r^3+\alpha^3\right)^{\frac{1}{3}}, \quad 0\le r<\infty,$$

$$\alpha=const.$$

Note that (2) is singular only when $r=0$ (in which case the metric does not actually apply), and that the constant α is indeterminable (Schwarzschild did not set $\alpha=2m$ for this reason).

For a 2-D spherically symmetric geometric surface [19] determined by

$$ds^2=R_c^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{3}$$

$$R_c=R_c(r),$$

the Riemannian curvature (which depends upon direction) reduces to the Gaussian curvature K (which is independent of direction), given by [13, 14, 20–23],

$$K = \frac{R_{1212}}{g},$$

where $R_{ijkm} = g_{in} R^{n}_{\cdot jkm}$ is the Riemann tensor of the first kind and $g = g_{11} g_{22} = g_{\theta\theta} g_{\varphi\varphi}$. Recall that

$$R^{1}_{\cdot 212} = \frac{\partial \Gamma^{1}_{22}}{\partial x^{1}} - \frac{\partial \Gamma^{1}_{21}}{\partial x^{2}} + \Gamma^{k}_{22}\Gamma^{1}_{k1} - \Gamma^{k}_{21}\Gamma^{1}_{k2},$$

$$\Gamma^{\alpha}_{\alpha\beta} = \Gamma^{\alpha}_{\beta\alpha} = \frac{\partial}{\partial x^{\beta}}\left(\frac{1}{2}\ln|g_{\alpha\alpha}|\right),$$

$$\Gamma^{\alpha}_{\beta\beta} = -\frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\beta\beta}}{\partial x^{\alpha}}, \quad (\alpha \neq \beta),$$

and all other $\Gamma^{\alpha}_{\beta\gamma}$ vanish. In the above, $k, \alpha, \beta = 1, 2$, $x^1 = \theta$ and $x^2 = \phi$, of course.

Straightforward calculation gives for expression (3),

$$K = \frac{1}{R_c^2},$$

so that R_c is the inverse square root of the Gaussian curvature, i. e. the radius of Gaussian curvature, and so r in Hilbert's "Schwarzschild's solution" is the radius of Gaussian curvature. The geodesic (i.e. proper) radius, R_p, of Schwarzschild's solution, up to a constant of integration, is given by

$$R_p = \int \frac{dR(r)}{\sqrt{1 - \frac{\alpha}{R(r)}}}, \tag{4}$$

and for Hilbert's "Schwarzschild's solution", by

$$R_p = \int \frac{dr}{\sqrt{1 - \frac{2m}{r}}}.$$

Thus the proper radius and the radius of Gaussian curvature *are not the same*; for the above, in general, $R_p \neq R(r)$ and $R_p \neq r$ respectively. The radius of Gaussian curvature does not determine the geodesic radial distance from the centre of spherical symmetry of the metric manifold. It is a "radius" only in the sense of it being the inverse square root of the Gaussian curvature. A detailed development of the foregoing, from first principles, is given in [13] and [14].

Note that in (2), if $\alpha = 0$ Minkowski space is recovered:

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$0 \leq r < \infty.$$

In this case the radius of Gaussian curvature is r and the proper radius is

$$R_p = \int_0^r dr = r,$$

so that the radius of Gaussian curvature and the proper radius are identical. It is for this reason that the radii, great circumferences, surface areas and volumes of spheres, etc., in Minkowski space can be determined in terms of the radius of Gaussian curvature. However, in the case of a (pseudo-) Riemannian manifold, such as (1) and (2) above, only great circumferences and surface areas can be determined via the radius of Gaussian curvature. Distances from the centre of spherical symmetry to a geodesic spherical surface in a Riemannian metric manifold can only be determined via the proper radius, except for particular points (if any) in the manifold where the radius of Gaussian curvature and the geodesic radius are identical, and volumes by a triple integral involving a function of the radius of Gaussian curvature. In the case of Schwarzschild's solution (2) (and hence also for (1)), the radius of Gaussian curvature, $R_c = R(r)$, and the proper radius, R_p, are identical only at $R_c \approx 1.467\alpha$. When the radius of Gaussian curvature, R_c, is greater than $\approx 1.467\alpha$, $R_p > R_c$, and when the radius of Gaussian curvature is less than $\approx 1.467\alpha$, $R_p < R_c$.

The upper and lower bounds on the Gaussian curvature (and hence on the radius of Gaussian curvature) are not arbitrary, but are determined by the proper radius in accordance with the intrinsic geometric structure of the line-element (which completely determines the geometry), manifest in the integral (4). Thus, one cannot merely assume that the radius of Gaussian curvature for (1) and (2) can vary from zero to infinity. Indeed, in the case of (2) (and hence also of (1)), as R_p varies from zero to infinity, the Gaussian curvature varies from $1/\alpha^2$ to zero and so the radius of Gaussian curvature correspondingly varies from α to infinity, as easily determined by evaluation of the constant of integration associated with the indefinite integral (4). Moreover, in the same way, it is easily shown that expressions (1) and (2) can be generalised [16] to all real values, but one, of the variable r, so that both (1) and (2) are particular cases of the general radius of Gaussian curvature, given by

$$R_c = R_c(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{1}{n}}, \tag{5}$$

$$r \in \Re, \quad n \in \Re^+, \quad r \neq r_0,$$

wherein r_0 and n are entirely arbitrary constants. Choosing $n = 3$, $r_0 = 0$ and $r > r_0$ yields Schwarzschild's solution (2). Choosing $n = 1$, $r_0 = \alpha$ and $r > r_0$ yields line-element (1) as determined by Johannes Droste [24] in May 1916, independently of Schwarzschild. Choosing $n = 1$, $r_0 = \alpha$ and $r < r_0$ gives $R_c = 2\alpha - r$, with line-element

$$ds^2 = \left(1 - \frac{\alpha}{2\alpha - r}\right)dt^2 - \left(1 - \frac{\alpha}{2\alpha - r}\right)^{-1}dr^2 - (2\alpha - r)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Using relations (5) directly, all real values of $r \neq r_0$ are permitted. In any case, however, the related line-element is singular only at the arbitrary parametric point $r = r_0$ on the real line (or half real line, as the case may be), which is the only parametric point on the real line (or half real line, as the case may be) at which the line-element fails (at $R_p(r_0) = 0 \ \forall\, r_0 \ \forall\, n$).

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