CARTAN GEOMETRY AND THE GYROSCOPE.

by

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ABSTRACT

It is shown that rotational dynamics is governed in general by the Cartan covariant derivative whose spin connection is a well define matrix of angular velocities. This matrix can be expressed in terms of any set of coordinates, and also in terms of the Euler angles. Rotational dynamics is therefore a sub structure of Cartan geometry, and can be extended in many different ways using the principles of Cartan geometry and ECE2 theory. Examples are given of the application of the theory to various types of gyroscope motion.

Keywords ECE2 theory, Cartan geometry, rotational dynamics, gyroscope motions.

UFT 370

1. INTRODUCTION

In recent papers and books of this series {1 - 12} the theory of the gyroscope has been developed in various ways and applied to explain the well known Laithwaite experiment in which a gyroscope held at arm's length appears to be weightless. In Section 2 of this paper it is shown that rotational dynamics in general is an example of the Cartan covariant derivative with well defined spin connection, and so rotational dynamics is a well defined limit of ECE2 theory. This result is exemplified by applications to the rotational dynamics of the asymmetric top using Euler angles and spherical polar coordinates. The transformation from spherical polar coordinates to Euler angles is defined. The use of spherical polar coordinates simplifies the development of the dynamics of a gyroscope with one point fixed, and allows the relevant torque to be well defined. Having attained this understanding the effect of an additional torque can be investigated. Laithwaite used an additional torque to lift the gyroscope and this torque can be modelled by computer. A dumb bell model is developed of the earth, treated as a gyroscope in orbit around the sun. This theory gives the basics of the Milankovitch cycles. Finally the translational kinetic energy is developed in terms of the Euler angles and spherical polar coordinates, a theory that can be used for spherical orbits and for the description of the nutations and precessions of the earth in orbit.

In Section 3, a dumb bell model is abalyzed and developed numerically, with graphics of the motion. Example graphics are given of solutions to the problems solved in Section 2. The key advance is the use of Maxima code to solve the relevant sets of Euler Lagrange equations.

This paper is a synopsis of extensive calculations contained in the notes accompanying UFT370 on www.aias.us and www.upitec.org. Note 370(1) contains some remarks on the methods used in UFT369. Note 370(2) is a first theory of the gyroscope subjected to an external torque. Note 370(3) is the basis for part of Section 2 and develops the general theory of rotational dynamics in terms of Cartan geometry. Notes 370(4) and 370(5) give the general theory the rotational dynamics of the asymmetric top in terms both of Euler angles and spherical polar coordinates. Note 370(6) gives the general relation between the angles of the spherical polar system and the Euler angles. Note 370(7) develops the theory of the gyroscope with one point fixed in terms of the spherical polar coordinates, and defines the relevant laboratory frame torque. Having defined the torque in this way , it becomes clear how to apply and model the Laithwaite torque by computer. The use of spherical polar coordinates is much simpler than the use of Euler angles, so the former method is preferred by Ockham's Razor. Note 370(8) is an extension of the dumb bell model developed in Section 3, and Note 370(9) defines the translational kinetic energy in terms of spherical polar coordinates and Euler angles, a definition needed for the understanding of Milankovitch cycles.

2. ROTATIONAL DYNAMICS AS CARTAN GEOMETRY

Rotational motion is developed from first principles in Note 370(3) which should be read with this synopsis. These principles are well known, but are given in all detail in Note 370(3) for clarity of exposition. They result in the well known theorem for the time derivative of any vector $\underline{F}$:

$$\left(\frac{d\underline{F}}{dt}\right)_{XYZ} = \left(\frac{d\underline{F}}{dt} + \underline{\omega} \times \underline{F}\right)_{123} . \quad -(1)$$

The left hand side of this equation is the derivative in the laboratory frame (X, Y, Z), and the right hand side is the derivative in the rotating frame (1, 2, 3). Here, $\underline{\omega}$ is the angular velocity vector defined in frame (1, 2, 3). An example of frame (1, 2, 3) is that of the principal moments of inertia of the rotating asymmetric top. In the Cartesian frame, often known as the inertial frame, the coordinate axes are not moving, but in frame (1, 2, 3) the coordinate axes are themselves rotating. Note 370(3) gives all details. Note carefully that Newtonian dynamics is defined in the inertial frame. In frame (1, 2, 3), for example of a gyroscope, more terms appear such as the centrifugal and Coriolis forces.

Eq. (1) can be written as:

$$\frac{d}{dt}\begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \quad -(2)$$

where ω_1, ω_2 and ω_3 are components of the angular velocity in frame (1, 2, 3). Eq. (2) is a special case of the Cartan covariant derivative:

$$\frac{DF^a}{\partial x^\mu} = \frac{\partial F^a}{\partial x^\mu} + \Omega^a{}_{\mu b} F^b \quad -(3)$$

where $\Omega^a{}_{\mu b}$ denotes the Cartan spin connection {1 - 12}. Considering the components:

$$\left.\begin{array}{l} \mu = 0, \\ a = 1, 2, 3 \\ b = 1, 2, 3 \end{array}\right\} \quad -(4)$$

it follows that:

$$\frac{DF^1}{\partial t} = \frac{\partial F^1}{\partial t} + \Omega^1{}_{01} F^1 + \Omega^1{}_{02} F^2 + \Omega^1{}_{03} F^3 \quad -(5)$$

so:

$$\frac{d}{dt}\begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} + \begin{bmatrix} \Omega^1{}_{01} & \Omega^1{}_{02} & \Omega^1{}_{03} \\ \Omega^2{}_{01} & \Omega^2{}_{02} & \Omega^3{}_{02} \\ \Omega^3{}_{01} & \Omega^3{}_{02} & \Omega^3{}_{03} \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \quad -(6)$$

The spin connection matrix of Cartan for rotational dynamics is therefore:

$$\begin{bmatrix} \Omega^1{}_{01} & \Omega^1{}_{02} & \Omega^1{}_{03} \\ \Omega^2{}_{01} & \Omega^2{}_{02} & \Omega^2{}_{03} \\ \Omega^3{}_{01} & \Omega^3{}_{02} & \Omega^3{}_{03} \end{bmatrix} := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad -(7)$$

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An example of Eq. (1) is the torque defined by:

$$\underline{Tq}_{XYZ} = \left(\frac{d\underline{L}}{dt}\right)_{XYZ} = \left(\frac{d\underline{L}}{dt} + \underline{\omega} \times \underline{L}\right)_{123} \quad -(8)$$

where $\underline{L}$ is the angular momentum. Using:

$$L_i = I_i \omega_i \quad -(9)$$

where I_i are the principal moments of inertia for:

$$i = 1, 2, 3 \quad -(10)$$

Eq. (8) becomes Euler's equations in frame (1, 2, 3)

$$Tq_1 = I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \quad -(11)$$
$$Tq_2 = I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \quad -(12)$$
$$Tq_3 = I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 . \quad -(13)$$

The Cartan spin connection for the Euler equations is therefore given by Eq. (7):

$$\Omega^a{}_{0b} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} . \quad -(14)$$

The torque vector is defined by:

$$\begin{aligned} \underline{Tq} &= Tq_X \underline{i} + Tq_Y \underline{j} + Tq_Z \underline{k} \\ &= Tq_1 \underline{e}_1 + Tq_2 \underline{e}_2 + Tq_3 \underline{e}_3 \end{aligned} \quad -(15)$$

so:

$$Tq_x^2 + Tq_y^2 + Tq_z^2 = Tq_1^2 + Tq_2^2 + Tq_3^2 \quad -(16)$$

These torque equations are true in general. In terms of the Euler angles θ, ϕ, and χ the angular velocities are given by:

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\chi}_1 = \dot{\phi}\sin\theta\sin\chi + \dot{\theta}\cos\chi \quad -(17)$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\chi}_2 = \dot{\phi}\sin\theta\cos\chi - \dot{\theta}\sin\chi \quad -(18)$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\chi}_3 = \dot{\phi}\cos\theta + \dot{\chi} \quad -(19)$$

The relevant lagrangian is developed in Note 370(4) and is:

$$\mathcal{L} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - U(\underline{r}, \theta, \phi, \chi) \quad -(20)$$

with Lagrange variables $\underline{r}$, θ, ϕ, and χ. The lagrangian is a sum of the translational kinetic energy of the centre of mass of the asymmetric top:

$$T_{trans.} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} \quad -(21)$$

its rotational kinetic energy:

$$T_{rot} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right). \quad -(22)$$

and the potential energy

$$U = U(\underline{r}, \theta, \phi, \chi) \quad -(23)$$

which is general a function of all four Lagrange variables. The dynamics are defined completely by the four Euler Lagrange equations:

$$\underline{\nabla}\,\mathcal{L} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(24)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(25)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(26)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad -(27)$$

which can be solved by Maxima as in the immediately preceding papers.

In terms of spherical polar coordinates (Note 370(5)) the spin connection matrix is:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta}_1 & -\dot{\phi}_1 \sin\theta_1 \\ \dot{\theta}_1 & 0 & -\dot{\phi}_1 \cos\theta_1 \\ \dot{\phi}_1 \sin\theta_1 & \dot{\phi}_1 \cos\theta_1 & 0 \end{bmatrix} \quad -(28)$$

where the subscript 1 is used to distinguish the angles of the spherical polar coordinate system from the Euler angles. So:

$$\omega_1 = \dot{\phi}_1 \cos\theta_1 \quad -(29)$$

$$\omega_2 = -\dot{\phi}_1 \sin\theta_1 \quad -(30)$$

$$\omega_3 = \dot{\theta}_1 \quad -(31)$$

and the angular velocity vector in frame (1, 2, 3) is:

$$\underline{\omega} = \dot{\phi}_1 \cos\theta_1\, \underline{e}_r - \dot{\phi}_1 \sin\theta_1\, \underline{e}_\theta + \dot{\theta}_1\, \underline{e}_\phi . \quad -(32)$$

The rotational kinetic energy of a freely rotating asymmetric top is therefore:

$$\mathcal{L} = T_{rot} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

$$= \frac{1}{2}\left(I_1 \dot{\phi}_1^2 \cos^2\theta_1 + I_2 \dot{\phi}_1^2 \sin^2\theta_1 + I_3 \dot{\theta}_1^2\right) \quad -(33)$$

and the simultaneous solution of the two Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\right) \quad -(34)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) \quad -(35)$$

gives the trajectories $\phi_1(t), \theta_1(t)$ and the angular velocities $\dot{\theta}_1, \dot{\phi}_1$ of the asymmetric top. Some calculational details are given in Note 370(5), and the algebra can be worked out completely by Maxima.

Therefore the angular velocity vector in frame (1, 2, 3) expressed in the spherical polar and Eulerian systems is:

$$\underline{\omega} = \omega_r \underline{e}_r + \omega_\theta \underline{e}_\theta + \omega_\phi \underline{e}_\phi \quad -(36)$$
$$= \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3$$

and it follows that:

$$\dot{\phi}_1^2 + \dot{\theta}_1^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}\left(2\dot{\phi}\cos\theta + \dot{\psi}\right) + 4\phi\theta\cos\psi\sin\psi\sin\theta \quad -(37)$$

This is the relation between angles of the spherical polar coordinates and the Euler angles in the rotating frame (1, 2, 3).

Both the Euler angles and the spherical polar coordinates should be used in general to extract all the dynamical information for a given problem. Consider the asymmetric top gyroscope with one point fixed, so that the origins of frame (X, Y, Z,) and (1, 2, 3) are the same. There is no translational kinetic energy because the point is fixed, so the lagrangian in terms of Euler angles is:

$$\mathcal{L} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - mgh\cos\theta$$
$$= \frac{1}{2}\left(I_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2\right.$$

$$+ I_2 \left(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \right)^2$$
$$+ I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2 \Bigg) - mgh\cos\theta \quad -(38)$$

in which the potential energy is:

$$U = mgh\cos\theta \quad -(39)$$

where h is the distance from the origin to the centre of mass along a principal moment of inertia axis. The latter is inclined at an angle θ to the Z axis of the laboratory frame. Here m is the mass of the gyroscope and g is the acceleration due to gravity. The Lagrange variables are the Euler angles, and there are three Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(40)$$
$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(41)$$
$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad -(42)$$

which must be solved simultaneously to give the trajectories $\theta(t), \phi(t), \psi(t)$ and the angular velocities: $\dot{\theta}, \dot{\phi}, \dot{\psi}$ which define the nutations and precessions of the gyroscope.

In the spherical polar representation of frame (1, 2, 3) the same lagrangian is:

$$\mathcal{L} = \frac{1}{2} \left(I_1 \dot{\phi}_1^2 \cos^2\theta_1 + I_2 \dot{\phi}_1^2 \sin^2\theta_1 + I_3 \dot{\theta}_1^2 \right) - mgh\cos\theta_1 \quad -(43)$$

and there are only two Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) \quad -(44)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\right) \quad -(45)$$

which can be solved for the trajectories $\theta_1(t), \phi_1(t)$ and angular velocities $\dot{\theta}_1$, $\dot{\phi}_1$. These define the precessions and nutations in another way. The numerical solutions must be checked for consistency using Eq. (37).

The angular velocity in general is:

$$\underline{\omega} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 + \omega_3 \underline{e}_3 \quad -(46)$$

in the frame (1, 2, 3) of the principal moments of inertia. The angular momentum in this frame is:

$$\underline{L} = I_1 \omega_1 \underline{e}_1 + I_2 \omega_2 \underline{e}_2 + I_3 \omega_3 \underline{e}_3 \quad -(47)$$

and the torque components in this frame are given by the Euler angles as in Eqs. (11) to (13). In general, the torque in the laboratory frame is:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(48)$$

where $\underline{r}$ is the position vector from the origin to the point where the force $\underline{F}$ is applied. The force is defined by:

$$\underline{F} = -\underline{\nabla} U \quad -(49)$$

where U is the potential energy. The laboratory frame force that must be applied to give a potential energy of the type (39) is:

$$\underline{F} = -\underline{\nabla} U = -\left(\frac{\partial U}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\underline{e}_\phi\right) \quad -(50)$$

in the spherical polar coordinates of the laboratory frame. Therefore the force is:

$$\underline{F} = \frac{mgh}{r} \sin\theta \, \underline{e}_\theta \quad -(51)$$

and the laboratory frame torque that must be applied to give the potential energy (39) is:

$$\underline{Tq} = \underline{r} \times \underline{F} \quad -(52)$$

where:

$$\underline{r} = r \, \underline{e}_r \quad -(53)$$

in the spherical polar coordinate system of the laboratory frame. Therefore the torque that must be applied is:

$$\underline{Tq} = mgh \sin\theta \, \underline{e}_r \times \underline{e}_\theta \quad -(54)$$

where the unit vectors of the spherical polar coordinate system of the laboratory frame are defined by:

$$\left.\begin{aligned} \underline{e}_\phi \times \underline{e}_r &= \underline{e}_\theta \\ \underline{e}_\theta \times \underline{e}_\phi &= \underline{e}_r \\ \underline{e}_r \times \underline{e}_\theta &= \underline{e}_\phi \end{aligned}\right] . \quad -(55)$$

So the laboratory frame torque is:

$$\underline{Tq} = mgh \sin\theta \, \underline{e}_\phi . \quad -(56)$$

Converting to Cartesian coordinates:

$$\underline{e}_\phi = -\underline{i} \sin\phi + \underline{j} \cos\phi \quad -(57)$$

so the laboratory frame torque needed to produce the potential enerrgy (39) is:

$$\underline{Tq} = mgl\sin\theta\left(-\underline{i}\sin\phi + \underline{j}\cos\phi\right). \quad -(58)$$

By definition:

$$\left.\begin{aligned} X &= l\sin\theta\cos\phi \\ Y &= l\sin\theta\sin\phi \end{aligned}\right\} \quad -(59)$$

so the laboratory frame torque is:

$$\underline{Tq} = mg\left(-Y\underline{i} + X\underline{j}\right) \quad -(60)$$

with:

$$\underline{r} = r\,\underline{e}_r \quad -(61)$$

and

$$\underline{F} = m\underline{g} = -mg\underline{k}. \quad -(62)$$

Note carefully that the attractive force due to gravity is in the negative k direction. Therefore:

$$\underline{Tq} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ X & Y & 0 \\ 0 & 0 & -mg \end{vmatrix} = mg\left(-Y\underline{i} + X\underline{k}\right) \quad -(63)$$

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The torque balance equation is therefore:

$$\underline{Tq} = mg\left(-Y\underline{i} + X\underline{j}\right) = Tq_1\,\underline{e}_1 + Tq_2\,\underline{e}_2 + Tq_3\,\underline{e}_3 \quad -(64)$$

and the gyroscope does not fall over as in the simple spinning top. Having exemplified the mathematics in detail as above, any additional laboratory frame torque may now be applied and the Laithwaite experiment may be simulated.

Two further examples are given in Notes 370(8) and 370(9) of the application of Cartan geometry to rotational dynamics: a dumbbell model of the earth's orbit around the sun, the earth being modelled by a dumbbell gyroscope, and in Note 370(9) the translational kinetic energy of a gyroscope is worked out in terms of the Euler angles and the spherical polar coordinates.

3. NUMERICAL SOLUTIONS AND GRAPHICS.

Section by Dr. Horst Eckardt

Cartan geometry and the gyroscope

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3 Numerical solution and graphics

We consider an example where the rotation is desribed by two mass points with a fixed coupling of distance $2h$, a so-called dumbbell-model, see Fig. 1. There is a spherical coordinate system (θ, ϕ) with origin in the middle of the connection of the two masses (red). This is the centre of mass which rotates around a central mass (blue) with another set of spherical polar coordinates (θ_1, ϕ_1, r). The coordinate transformation of the two masses to cartesian coordinates relative to the centre of mass is

$$\mathbf{h}_1 = h \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \quad \mathbf{h}_2 = -\mathbf{h}_1 \tag{65}$$

and the coordinate $\mathbf{R}$ of the centre of mass is

$$\mathbf{R} = r \begin{bmatrix} \sin\theta_1\cos\phi_1 \\ \sin\theta_1\sin\phi_1 \\ \cos\theta_1 \end{bmatrix}. \tag{66}$$

The Lagrange formalism requires the coordinates of both masses in the global cartesian system:

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{h}_1, \tag{67}$$
$$\mathbf{r}_2 = \mathbf{R} + \mathbf{h}_2. \tag{68}$$

Their kinetic energy is

$$E_{kin} = \frac{1}{2} m \left(\dot{\mathbf{r}}_1\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2\dot{\mathbf{r}}_2\right). \tag{69}$$

For the potential energy we make an approximation which holds when the two masses stay far from the gravitational centre which is the case for planets for

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example. Instead of using $\mathbf{r}_1$ and $\mathbf{r}_2$ we insert r, obtaining the sum of potential energy of both masses:

$$E_{pot} = -2\frac{mMG}{r}. \tag{70}$$

The Lagrangian then takes the simple form

$$\mathscr{L} = m\left(r^2(\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin(\theta_1)^2) + h^2(\dot{\theta}^2 + \dot{\phi}^2 \sin(\theta)^2)\right) + 2\frac{mMG}{r} \tag{71}$$

and the Euler-Lagrange equations according to (34,35) for the five Lagrangian variables are:

$$\ddot{\theta} = \dot{\phi}^2 \cos(\theta) \sin(\theta), \tag{72}$$

$$\ddot{\phi} = -\frac{2\dot{\theta}\dot{\phi}\cos(\theta)}{\sin(\theta)}, \tag{73}$$

$$\ddot{\theta_1} = -\frac{2\dot{r}\dot{\theta}_1 - \dot{\phi_1}^2 r\cos(\theta_1)\sin(\theta_1)}{r}, \tag{74}$$

$$\ddot{\phi_1} = -\frac{2r\dot{\theta}_1\dot{\phi}_1\cos(\theta_1) + 2\dot{r}\dot{\phi}_1\sin(\theta_1)}{r\sin(\theta_1)}, \tag{75}$$

$$\ddot{r} = r\dot{\theta_1}^2 + r\dot{\phi_1}^2\sin(\theta_1)^2 - \frac{MG}{r^2}. \tag{76}$$

Numerical solution with the Maxima code gives the results shown in Figs. 2-4. Obviously the central coordinates (θ_1, ϕ_1, r) decouple from those local to the dumbbell masses (θ, ϕ). The latter show oscillations of nutation and precession, see Fig. 2. From the trajectory graph of the central coordinates (Fig. 3) can be seen that θ_1 stays at its initial value of $\pi/2$, i.e. the motion takes place in a plane and is not distorted by the dumbbell. The radius oscillates between a maximum value and the half of it, it is an elliptic orbit. Correspondingly, the azimuthal angle ϕ varies with different velocities, they are higher at the perihelion as is well known from motion of planets.

In Fig. 4 the orbits of the centre of mass (blue) and one of the dumbbell masses (red) are graphed. The central planar motion can be seen which is overlaid with a three-dimensional oscillation of the masses. This may serve as a simple model for the Milankowitch cycles. The latter are very slow compared to one orbit, here we have chosen the parameters in a way that the deviations from the ellipse can be seen easily.

As another example we solve the motion of a rotating rigid body in spherical polar coordinates as described by Eqs. (33-35). This leads to the equations of motion

$$\ddot{\theta_1} = \frac{(I_2 - I_1)\dot{\phi_1}^2\cos(\theta_1)\sin(\theta_1)}{I_3}, \tag{77}$$

$$\ddot{\phi_1} = -\frac{2(I_2 - I_1)\dot{\theta}_1\dot{\phi}_1\cos(\theta_1)\sin(\theta_1)}{I_1\cos(\theta_1)^2 + I_2\sin(\theta_1)^2}. \tag{78}$$

It is seen that these equations transform to free motion in the case $I_1 = I_2$, i.e. the right hand sides become zero. A symmetric top rotates with constant angular velocity. The equations have been solved for $I_1 = 1,\ I_2 = 1.5,\ I_3 = 2.5$

and the solutions are graphed in Fig. 5. The polar angle θ describes a nutation and ϕ increases irregularly. The reason is the oscillating behaviour of the angular velocity vector $\boldsymbol{\omega}$ whose components are graphed in Fig. 6. The modulus of $\boldsymbol{\omega}$ is not constant, this is not a constant of motion.

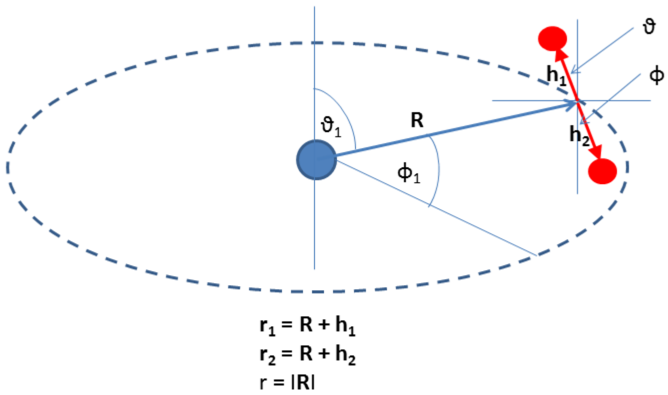

Figure 1: The rotating dumbbell model with coordinates.

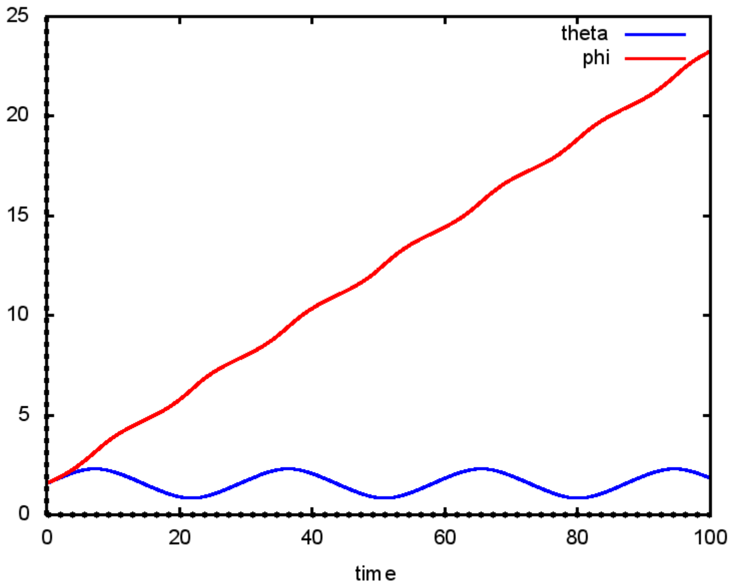

Figure 2: Rotating dumbbell, trajectories $\theta(t)$ $\phi(t)$.

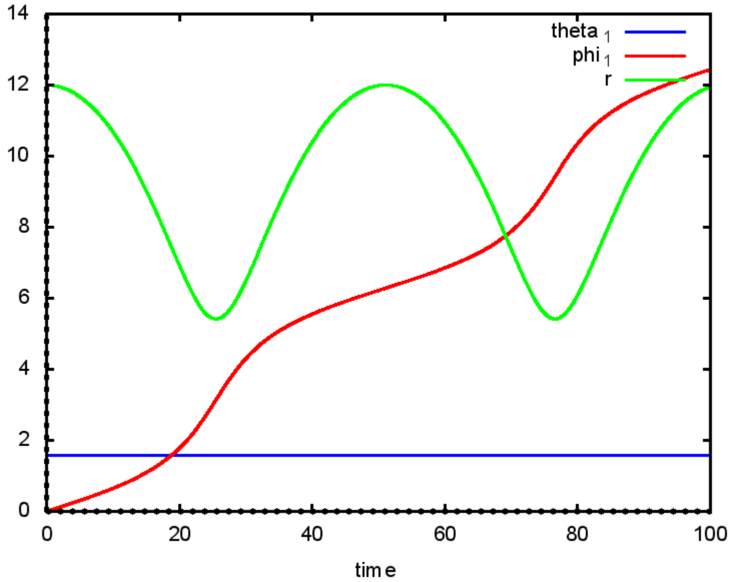

Figure 3: Motion of centre of mass, trajectories $\theta_1(t)$, $\phi_1(t)$, $r(t)$.

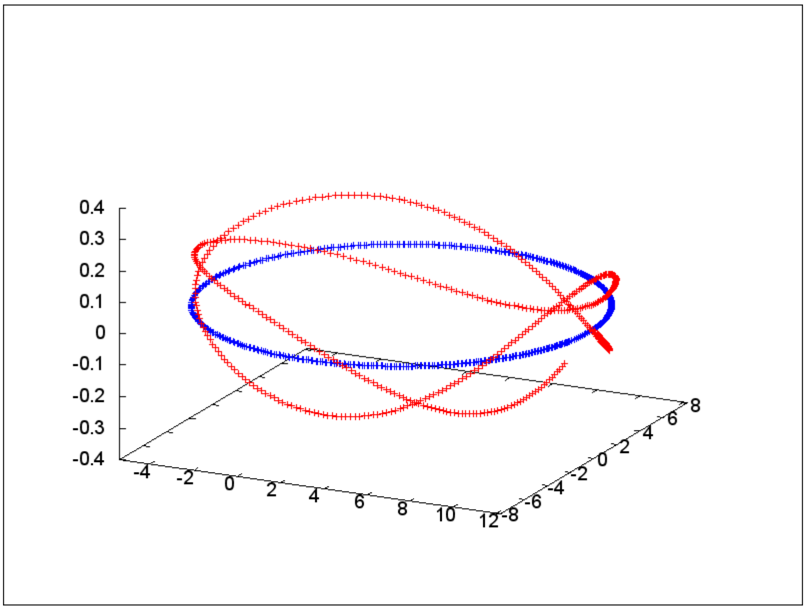

Figure 4: Orbit of the centre of mass (blue) and one dumbbell mass (red).

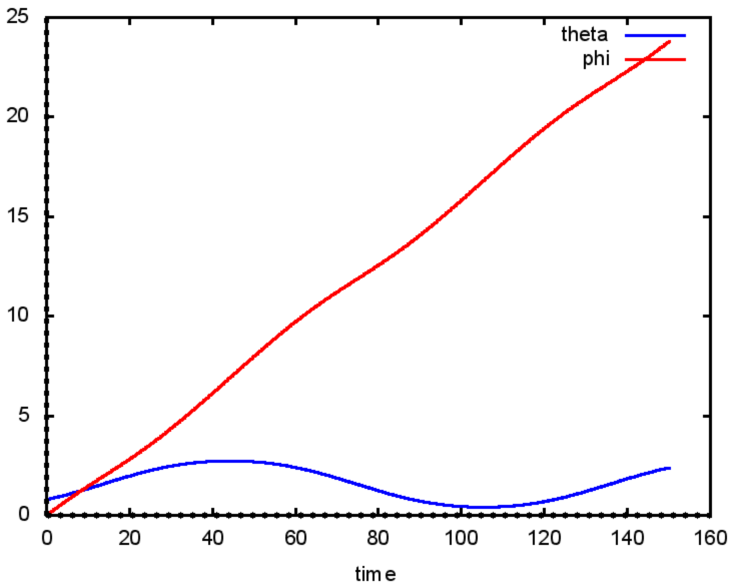

Figure 5: Rotating rigid body, trajectories $\theta_1(t)$ $\phi_1(t)$ in spherical coordinates.

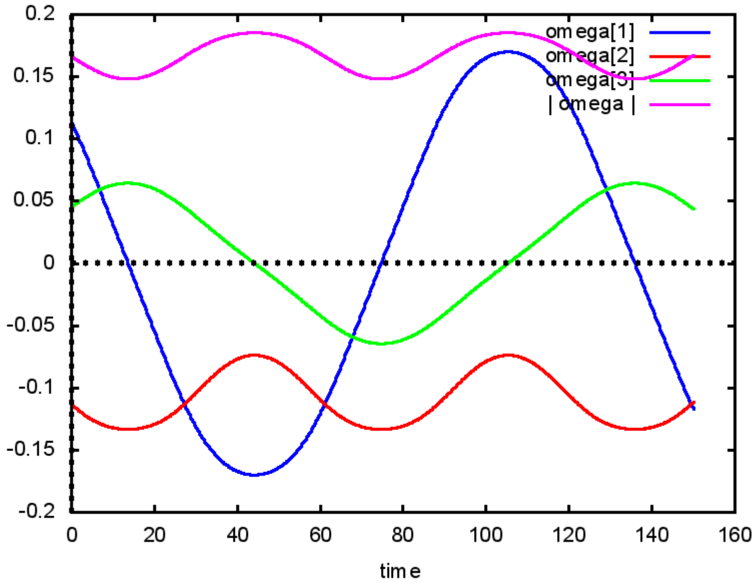

Figure 6: Rotating rigid body, angular velocities $\omega_{1,2,3}$ and modulus of $\boldsymbol{\omega}$.

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