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Rank Three Tensors in Unified Gravitation and Electrodynamics

by

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Abstract

The role of base manifold rank three tensors is discussed in generally covariant unified field theory. It is shown that the rank three tensor in a space-time with torsion and curvature is in general dual to a rank one four-vector, a duality that follows from fundamental considerations of general coordinate transformation within the Einstein group. When considering the rotation group, a sub-group of the Einstein group, the rank three tensor is anti-symmetric in its lower two indices. Examples include the canonical angular energy-momentum density tensor, the Cartan torsion tensor, and the electromagnetic field tensor. The field equations of classical electrodynamics are expressed in terms of such a rank three tensor and reduced to the familiar Maxwell Heaviside vector format with the inclusion of the spin connection.

Keywords: Generally covariant unified field theory, rank three tensors, duality, general coordinate transformation, field equations of classical electrodynamics.

5.1 Introduction

In the Maxwell Heaviside field theory of classical electrodynamics [1] it is well known that the electromagnetic field tensor is a rank two tensor in the Minkowski space-time. This is a flat space-time with no torsion and no curvature and is covariant under the Lorentz transformation. As such it cannot be unified in a meaningful manner with the generally covariant Riemannian space-time of the classical theory of gravitation [2]. Recently [3–12] a generally covariant unified field theory has been proposed based on standard differential geometry in a space-time with torsion and curvature present simultaneously in general. In this theory, known as Einstein Cartan Evans (ECE) field theory, all sectors are generally covariant, i.e. covariant under the general coordinate transformation as required by the philosophy of relativity. The electromagnetic field is within a primordial voltage the Cartan torsion form [2]. This is a vector valued differential two-form, an anti-symmetric tensor with an additional label a that comes from a tangent space-time at point P to a base manifold. Electromagnetic field equations have been deduced in ECE theory from the Bianchi identity and Cartan structure equations. It is shown in Section 5.2 that these field equations can be reduced to equations in the base manifold. The latter constitute a particular solution of the more general differential form equations and involve the use of a rank three electromagnetic field tensor in the base manifold, a space-time with non-zero torsion and curvature in general. In Section 5.3 it is shown that such a tensor is in general dual to a rank one four-vector in the base manifold, such a duality arises from considerations of the general coordinate transform in the Einstein group, of which the rotation group is a sub-group. For the rotation group the rank three tensor is anti-symmetric in its lower two indices, and components of such a rank three tensor can be identified with the components of a vector. In this way the ECE field equations can be reduced to vector equations which have the same structure as the familiar Maxwell Heaviside field equations but with the key addition of the spin connection. The ECE vector equations are written not in a Minkowski space-time, but in one with torsion and curvature, and the charge-current density is proportional to the Ricci tensor of Riemannian geometry. In Section 5.4 finally it is argued that the rank three Cartan torsion tensor in the base manifold is proportional to the canonical angular energy-momentum density from the Noether Theorem as well as being directly proportional to the rank three electromagnetic field tensor in ECE theory. ECE is therefore a fully consistent and generally covariant unified field theory whose origins are rigorously geometrical as required by the philosophy of relativity [2].

5.2 Field Equations in the Base Manifold

The homogeneous ECE equation [3–12] in differential form notation is:

$$D \wedge F^a = R^a{}_b \wedge A^b = A^b \wedge R^a{}_b \tag{5.1}$$

where F^a denotes the electromagnetic field form, $R^a{}_b$ the curvature form, A^b the electromagnetic potential form. The electromagnetic form is related to the electromagnetic tensor $F^{\kappa}_{\mu\nu}$ as follows:

$$F^a_{\mu\nu} = q^a_\kappa F^\kappa_{\mu\nu} \tag{5.2}$$

where the base manifold indices have been restored in Eq. (5.2). The left hand side of Eq. (5.1) is:

$$D \wedge F^a = D_\mu F^a_{\nu\sigma} + D_\sigma F^a_{\mu\nu} + D_\nu F^a_{\sigma\mu} \tag{5.3}$$

and the tetrad postulate is:

$$D_\mu q^a{}_\kappa = 0. \tag{5.4}$$

Therefore by Leibnitz's Theorem:

$$\begin{aligned} &D_\mu (q^a_\kappa F^\kappa{}_{\nu\sigma}) + D_\sigma (q^a{}_\kappa F^\kappa{}_{\mu\nu}) + D_\nu (q^a{}_\kappa F^\kappa{}_{\sigma\nu}) \\ &= q^a_\kappa (D_\mu F^\kappa{}_{\nu\sigma} + D_\sigma F^\kappa{}_{\mu\nu} + D_\nu F^\kappa{}_{\sigma\mu}). \end{aligned} \tag{5.5}$$

Similarly the right hand side of Eq. (5.1) is:

$$R^a{}_b \wedge A^b = A^{(0)} q^a{}_\kappa (R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} + R^\kappa{}_{\mu\nu\sigma}) \tag{5.6}$$

so:

$$q^a_\kappa (D_\mu F^\kappa{}_{\sigma\nu} + D_\sigma F^\kappa{}_{\mu\nu} + D_\nu F^\kappa{}_{\sigma\mu}) = q^a_\kappa A^{(0)} (R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} + R^\kappa{}_{\mu\nu\sigma}). \tag{5.7}$$

A particular solution of Eq. (5.7) is:

$$D_\mu F^\kappa{}_{\nu\sigma} + D_\sigma F^\kappa{}_{\mu\nu} + D_\nu F^\kappa{}_{\sigma\mu} = A^{(0)} (R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} + R^\kappa{}_{\mu\nu\sigma}). \tag{5.8}$$

By definition:

$$D_\mu F^\kappa{}_{\nu\sigma} = \partial_\mu F^\kappa{}_{\nu\sigma} + \omega^\kappa{}_{\mu b} F^b{}_{\nu\sigma} \tag{5.9}$$

where $\omega^\kappa{}_{\mu b}$ is the appropriate form of the spin connection. The structure of Eq. (5.8) is therefore:

$$\partial_\mu F^\kappa{}_{\nu\sigma} + \partial_\sigma F^\kappa{}_{\mu\nu} + \partial_\nu F^\kappa{}_{\sigma\mu} = \frac{A^{(0)}}{\mu_0} (j^\kappa{}_{\mu\nu\sigma} + j^\kappa{}_{\sigma\mu\nu} + j^\kappa{}_{\nu\sigma\mu}). \tag{5.10}$$

For all practical purposes [3–12] the homogeneous current is zero, so Eq. (5.10) is:

$$\partial_\mu \widetilde{F}^{\kappa\mu\nu} = 0 \tag{5.11}$$

Its Hodge dual [3–12] is:

$$\partial_\mu F^{\kappa\mu\nu} = -\frac{A^{(0)}}{\mu_0} R^{\kappa}{}_{\mu}{}^{\mu\nu}. \tag{5.12}$$

Here μ_0 is the vacuum permeability in S.I. units. It is seen that rank three tensors enter into the field equations (5.11) and (5.12). These are shown later in this paper to be in general dual to a four vector as follows:

$$V^\mu = V^\mu{}_{\rho\sigma}\, \epsilon^{\rho\sigma} \tag{5.13}$$

where for rotation [13]:

$$\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho}. \tag{5.14}$$

The rotation group is a sub group of the Lorentz group, which is in turn a sub group of the Poincaré group and of the Einstein group. In general therefore the electromagnetic field tensor is the anti-symmetric rank three tensor:

$$F^\mu{}_{\rho\sigma} = -F^\mu{}_{\sigma\rho}\,. \tag{5.15}$$

The Cartan torsion tensor is well known [2–12] to be:

$$\begin{aligned} T^\mu{}_{\rho\sigma} &= -T^\mu{}_{\sigma\rho} \\ &= \Gamma^\mu{}_{\rho\sigma} - \Gamma^\mu{}_{\sigma\rho} \end{aligned} \tag{5.16}$$

where $\Gamma^\mu{}_{\rho\sigma}$ is the general gamma connection. Another well known example of a rank three anti-symmetric tensor is the canonical angular momentum-energy density tensor [13]. These three tensors have the same fundamental anti-symmetry in their lower two indices and are shown later in this paper to be proportional to each other. Indeed, the fundamental ECE hypothesis in tensor notation is [3–12]:

$$F^\mu{}_{\rho\sigma} = A^{(0)} T^\mu{}_{\rho\sigma}\,. \tag{5.17}$$

Eq. (5.12) therefore has a clear interpretation in terms of a rank three tensor whose elements are electric and magnetic field three-vector components, and in terms of a Ricci type tensor $R^{\kappa}{}_{\mu}{}^{\mu\nu}$. The technical correctness of Eq. (5.12) has been demonstrated by computer algebra [3–12]. It has also been checked by computer algebra that in a Ricci flat space-time:

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} = 0 \tag{5.18}$$

meaning that:

$$\partial_\mu F^{\kappa\mu\nu} = 0 \tag{5.19}$$

self-consistently. Eq. (5.19) translates into the following two vector equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \tag{5.20}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mathbf{0} \tag{5.21}$$

which have the same vector structure as the vacuum Coulomb law and vacuum Ampère Maxwell law of Maxwell Heaviside (MH) field theory. However, Eqs. (5.20) and (5.21) of ECE theory are written in a space-time with torsion and curvature, not in a flat Minkowski space-time as in MH theory. In ECE theory the charge-current density is recognized as being

$$J^{\kappa\nu} = -\frac{A^{(0)}}{\mu_0} R^{\kappa}{}_{\mu}{}^{\mu\nu} \tag{5.22}$$

and to be proportional directly to a Ricci type tensor $R^{\kappa}{}_{\mu}{}^{\mu\nu}$. The vacuum in ECE theory is defined as being Ricci flat in a generally covariant unified field theory. This is self-consistently the vacuum solution of the Einstein Hilbert (EH) field equation (5.14).

5.3 General Coordinate Transformation and Rotation

The general coordinate transformation of a four-vector V^μ is defined [2] as:

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu. \tag{5.23}$$

The Lorentz transform is a special case of Eq. (5.23):

$$V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^\mu \tag{5.24}$$

and is a special kind of coordinate transformation [2]:

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu. \tag{5.25}$$

Eq (5.23) describes the behavior of vectors under arbitrary changes of coordinates and basis elements. The complete vector field is constant under the change of coordinates:

$$V = V^{\mu}\partial_{\mu} = V^{\mu'}\partial_{\mu'}. \tag{5.26}$$

A rotation is an example of a coordinate transformation, a rotation can be the rotation of a vector with fixed coordinates or the rotation of coordinates with fixed vector. Rotation in an arbitrary manifold is represented by the rotation group, which is a sub group of the Lorentz group. The latter is itself a subgroup of the Poincaré group [13], which is a subgroup of the Einstein group. Rotation generators are proportional to angular momentum generators [13] and are anti-symmetric tensors.

We now denote:

$$\nu = \mu' \tag{5.27}$$

and denote:

$$\epsilon^{\mu}{}_{\nu} = \partial x^{\mu}/\partial x^{\nu}. \tag{5.28}$$

Therefore:

$$\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho} = g^{\rho\kappa}\epsilon^{\sigma}{}_{\kappa} \tag{5.29}$$

where $g^{\rho\kappa}$ is the inverse metric tensor in an arbitrary manifold [2]. For rotations:

$$\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho} \tag{5.30}$$

and the Kronecker delta [2] is defined by:

$$\delta^{\mu}{}_{\sigma} = g^{\mu\nu}g_{\nu\sigma}. \tag{5.31}$$

It is seen that the coordinate four-vector can be represented by:

$$x^{\mu} = \epsilon^{\mu}{}_{\nu}x^{\nu} = X^{\mu}{}_{\rho\sigma}\epsilon^{\rho\sigma}. \tag{5.32}$$

This means that there exists a rank three tensor $X^{\mu}{}_{\rho\sigma}$ from considerations of coordinate transformation alone. For rotations:

$$X^{\mu}{}_{\rho\sigma} = -X^{\mu}{}_{\sigma\rho} \tag{5.33}$$

because:

$$\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho}. \tag{5.34}$$

Therefore there exists the following duality between V^μ and $V^\mu{}_{\rho\sigma}$:

$$V^\mu = V^\mu{}_{\rho\sigma}\,\epsilon^{\rho\sigma} \tag{5.35}$$

meaning that any four-vector V^μ can be expressed as a rank three tensor $V^\mu{}_{\rho\sigma}$ in the arbitrary base manifold. In 3-D Euclidean space there is a similar duality between an axial three-vector V_i and a rank two anti-symmetric tensor V_{jk}:

$$V_i = \frac{1}{2}\epsilon_{ijk}V_{jk} \tag{5.36}$$

where ϵ_{ijk} is the rank three totally anti-symmetric unit tensor in three dimensional Euclidean space.

In general, Eq. (5.35) is:

$$\frac{1}{2}V^\mu = V^\mu{}_{01}\,\epsilon^{01} + V^\mu{}_{02}\,\epsilon^{02} + V^\mu{}_{03}\,\epsilon^{03} + V^\mu{}_{12}\,\epsilon^{12} + V^\mu{}_{13}\,\epsilon^{13} + V^\mu{}_{23}\,\epsilon^{23}. \tag{5.37}$$

This can be expressed as a four dimensional matrix with a structure similar to the electromagnetic field matrix of MH theory [1] but with an additional upper index:

$$V^\mu{}_{\rho\sigma} = \begin{bmatrix} 0 & V^\mu{}_{01} & V^\mu{}_{02} & V^\mu{}_{03} \\ V^\mu{}_{10} & 0 & V^\mu{}_{12} & V^\mu{}_{13} \\ V^\mu{}_{20} & V^\mu{}_{21} & 0 & V^\mu{}_{23} \\ V^\mu{}_{30} & V^\mu{}_{31} & V^\mu{}_{32} & 0 \end{bmatrix}. \tag{5.38}$$

If for example rotation about the $Z = 3$ axis is considered, then:

$$\mu = 3, \epsilon^{12} = -\epsilon^{21} = 1 \tag{5.39}$$

all other $\epsilon^{\mu\nu}$ being zero, so:

$$V_Z = V^3 = 2V^3{}_{12} \tag{5.40}$$

Here V_Z is the Z component of the three-vector:

$$\boldsymbol{V} = V_X\boldsymbol{i} + V_Y\boldsymbol{j} + V_Z\boldsymbol{k}. \tag{5.41}$$

Eq. (5.40) is the magnetic field relation used in the ECE theory of classical electrodynamics to reduce it to the same vector format as the MH theory, but with the key addition of the spin connection [3–12]. The three axial vector relations are:

$$V_X = V^1 = 2V^1{}_{23} \tag{5.42}$$
$$V_Y = V^2 = 2V^2{}_{13} \tag{5.43}$$
$$V_Z = V^3 = 2V^3{}_{12} \,. \tag{5.44}$$

The three polar vector relations are:

$$V_X = V^1 = 2V^0{}_{01} \tag{5.45}$$
$$V_Y = V^2 = 2V^0{}_{02} \tag{5.46}$$
$$V_Z = V^3 = 2V^0{}_{03} \tag{5.47}$$

and define the electric field relations used in the ECE theory of electrodynamics [3–12]. The electromagnetic field tensor in ECE theory is directly proportional as follows to the Cartan torsion tensor:

$$F^\mu{}_{\sigma\rho} = A^{(0)} T^\mu{}_{\sigma\rho} \tag{5.48}$$

and both arise from the rotation generator in the arbitrary manifold. The Cartan torsion form is defined by the addition of a Minkowski space-time at a point P in the arbitrary base manifold. The Cartan torsion form is defined by the first Cartan structure equation [2]:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \tag{5.49}$$

and this definition is equivalent to:

$$T^\mu{}_{\sigma\rho} = \Gamma^\mu{}_{\sigma\rho} - \Gamma^\mu{}_{\rho\sigma} \tag{5.50}$$

using the tetrad postulate [2–12]:

$$D_\mu q^a{}_\nu = 0. \tag{5.51}$$

5.4 Canonical Angular Momentum-Energy Density

This kind of rank three tensor arises from considerations of the Noether Theorem [13], which is derived from a lagrangian method. It is also anti-symmetric in its lower two indices and is defined as:

$$J^{\mu}{}_{\rho\sigma} = -\frac{1}{2}(T^{\mu}{}_{\rho}\, x_{\sigma} - T^{\mu}{}_{\rho}\, x_{\rho}). \tag{5.52}$$

where:

$$T^{\mu\nu} = T^{\nu\mu} = g^{\nu\kappa} T^{\mu}{}_{\kappa} \tag{5.53}$$

is the symmetric canonical energy-momentum density tensor used in the EH field equation:

$$G_{\mu\nu} = kT_{\mu\nu} \tag{5.54}$$

where $G_{\mu\nu}$ is the Einstein tensor and k is the Einstein constant. From the Noether Theorem the action is invariant under spatial rotations:

$$\delta x^{i} = \epsilon^{ij} x^{j}, \epsilon^{ij} = -\epsilon^{ji} \qquad (i, j = 1, 2, 3) \tag{5.55}$$

as generalized in Section 5.3 of this paper and the canonical angular energy-momentum density:

$$J^{\mu\rho\sigma} = -T^{\mu}{}_{\kappa}\, X^{\kappa\rho\sigma} \tag{5.56}$$

is a conserved Noether current:

$$D_{\mu} J^{\mu}{}_{\rho\sigma} = 0. \tag{5.57}$$

This relation is analogous to the conservation of canonical energy-momentum density:

$$D^{\nu} T_{\mu\nu} = 0 \tag{5.58}$$

and the EH field equation was derived from:

$$\begin{aligned} D^{\mu} G_{\mu\nu} &= kD^{\nu} T_{\mu\nu} \\ &= 0 \end{aligned} \tag{5.59}$$

where the left hand side is the second Bianchi identity [2–12].

It is well known experimentally [15] that the electromagnetic field has angular momentum, and as we have argued the canonical form of this is ${J^{\mu}}_{\rho\sigma}$. So the latter is proportional to ${F^{\mu}}_{\rho\sigma}$, and the ECE hypothesis makes ${F^{\mu}}_{\rho\sigma}$ proportional to the Cartan torsion tensor ${T^{\mu}}_{\rho\sigma}$. All are rank three tensors anti-symmetric in their lower two indices and all derive from the rotation generator. These considerations give the results:

$$ {F^{\mu}}_{\rho\sigma} = \frac{c}{e\omega} {J^{\mu}}_{\rho\sigma} = \frac{E^{(0)}}{\omega} {T^{\mu}}_{\rho\sigma} \tag{5.60} $$

Using the quantum relation [3–12]:

$$ \left. \begin{aligned} eA^{(0)} &= \hbar\kappa, \\ E^{(0)} &= cB^{(0)} = \kappa C A^{(0)} = \omega A^{(0)} \end{aligned} \right\} \tag{5.61} $$

we obtain:

$$ {J^{\mu}}_{\rho\sigma} = \hbar\kappa^2 ({\Gamma^{\mu}}_{\rho\sigma} - {\Gamma^{\mu}}_{\sigma\rho}) \tag{5.62} $$

so that the anti-symmetric connection can be defined as:

$$ {\Gamma^{\mu}}_{\rho\sigma} = -\frac{1}{2\hbar\kappa^2} g^{\mu\kappa} T_{\kappa\rho} x_{\sigma}. \tag{5.63} $$

This is a self consistent result because in EH theory there is no angular momentum and no Cartan torsion, so that the connection in EH theory is the symmetric Christoffel connection [2]:

$$ {\Gamma^{\mu}}_{\rho\sigma} = {\Gamma^{\mu}}_{\sigma\rho}. \tag{5.64} $$

Therefore individual vector components of magnetic flux density and electric field strength may be defined as follows:

$$ E_X = {E^{0}}_{01} = (c^2/(e\omega)) {J^{0}}_{01} \tag{5.65} $$
$$ E_Y = {E^{0}}_{02} = (c^2/(e\omega)) {J^{0}}_{02} \tag{5.66} $$
$$ E_Z = {E^{0}}_{03} = (c^2/(e\omega)) {J^{0}}_{03} \tag{5.67} $$
$$ B_X = {B^{1}}_{23} = (c^2/(e\omega)) {J^{1}}_{23} \tag{5.68} $$
$$ B_Y = {B^{3}}_{12} = (c^2/(e\omega)) {J^{3}}_{12} \tag{5.69} $$
$$ B_Z = {B^{2}}_{31} = (c^2/(e\omega)) {J^{2}}_{31} \tag{5.70} $$

where:

$$E^0{}_{01} = \partial_0 A_1^0 - \partial_1 A_0^0 + \omega^0{}_{0b} A^b{}_1 - \omega^0{}_{1b} A^b{}_0 \tag{5.71}$$

$$B^1{}_{23} = \partial_2 A_3^1 - \partial_3 A_2^1 + \omega^1{}_{2b} A^b{}_3 - \omega^1{}_{3b} A^b{}_2 \tag{5.72}$$

and where:

$$J^0{}_{01} = -\frac{1}{2}(T^0{}_0\, x_1 - T^0{}_1\, x_0) \tag{5.73}$$

etc.

and:

$$J^1{}_{23} = -\frac{1}{2}(T^1{}_2\, x_3 - T^1{}_3\, x_2), \tag{5.74}$$

etc.

Acknowledgments

The British Government is thanked for the award of a Civil List Pension and the Telesio-Galilei Association for funding of this work.

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